GAME THEORY AND TRANSPORTATION SYSTEMS MODELLING

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Abstract—In this paper, correspondences are drawn between two game theory models (Nash noncooperative and Stackelberg games) and some problems in transportation systems modelling. An example of each is described in detail, namely the problem of carriers competing for intercity passenger travel and the signal optimization problem. The discussion serves to underline differences between two categories of transportation problems and introduces the game theory literature as a potential source of solution algorithms. Also it is shown that inner-outer iterative techniques for Stackelberg type problems cannot be expected to converge to the solution, and an approximate formulation of these problems is introduced which appears to be more readily solvable.

INTRODUCTION

The purpose of this paper is to present characteristics of two behavioral models from game theory which have application in decision-making models for planning and operating transportation systems. Specifically many such problems fit within the frameworks of a Stackelberg (Stackelberg, 1934) or a Nash noncooperative game (Nash, 1951). This study underlines the differences between these two games and gives examples of problems fitting into either framework. It serves to clarify differences between certain types of problems and by making explicit their correspondence with game theory, the range of potential solution techniques is broadened.

Game theory provides a framework for modelling interactions between groups of decision-makers when individual actions jointly determine the outcome. Concepts from this theory first appeared in transportation models in the form of a behavioral hypothesis for route choice, Wardrop's first principle (Wardrop, 1952), being identical to the statement characterizing a Nash noncooperative game, with each traveller corresponding to a player. It is shown in Zukhovitshii et al. (1973), that the Nash equilibrium conditions for an N-person noncooperative game are equivalent to those of a zero sum two person game, which in turn has an equivalent variational inequality (VI) formulation. Had this result been known to transportation researchers about the time it was proposed, an earlier discovery of a VI formulation of Wardrop's model would have been possible. (Smith derived this formulation of the problem in 1979).

Many other problems have equivalent representations as Nash noncooperative or Stackelberg games. These games are described in the next section for the two player case and problems fitting into either framework are discussed.

For illustrative purposes the signal optimization problem is studied in detail as a special case of a Stackelberg game and it is shown that a penalty approach can be used to simplify one formulation of this problem. It is also shown that inner-outer heuristic techniques for solving Stackelberg problems will not converge to the solution. A model of carriers competing for intercity passenger travel is studied as an example of a Nash noncooperative game.

NASH NONCOOPERATIVE AND STACKELBERG GAMES
WITH TWO PLAYERS

The objective of each player $i$, is to minimize a performance function $P_i(x_1, x_2)$ where $x_i$ ($i = 1, 2$) is the vector of decision variables for player $i$. In a Nash noncooperative game, the equilibrium state is characterized by the property that neither player can improve his
objective by unilaterally changing his decision. Thus if \((x_1^*, x_2^*)\) is the equilibrium solution, then

\[
\begin{align*}
P_1(x_1, x_2^*) & \geq P_1(x_1^*, x_2^*) \\
\frac{\partial P_1}{\partial x_2} & \leq \frac{\partial P_2}{\partial x_2}.
\end{align*}
\]

For a given strategy of the other player, \(i\)'s optimal strategy is found by solving

\[
\min_{x_i} P_i(x_1, x_2)
\]

and the equilibrium solution is the point at which the optimality conditions for (2) are satisfied simultaneously for \(i = 1, 2\). For example, suppose \(x_i \in \mathbb{R}\) and

\[
\begin{align*}
P_1(x_1, x_2) &= x_1^2 - x_1x_2 + 2x_2^2 + x_1 \\
P_2(x_1, x_2) &= 2x_1^2 - x_1x_2 + x_2^2,
\end{align*}
\]

then

\[
\begin{align*}
\frac{\partial P_1}{\partial x_1} &= 2x_1 - x_2 + 1 = 0 \\
\frac{\partial P_2}{\partial x_2} &= -x_1 + 2x_2 = 0.
\end{align*}
\]

The solution is

\[
\begin{align*}
x_1^* &= -\frac{2}{3} & P_1 &= -\frac{2}{9} \\
x_2^* &= -\frac{1}{3} & P_2 &= \frac{7}{9}.
\end{align*}
\]

In the Stackelberg game, one player (the leader) knows how the other player (the follower) will respond to any decision he may make. If player 1 is the leader and \(x_2 = T(x_1)\) is the response of player 2 to decision \(x_1\), then at equilibrium

\[
\begin{align*}
P_1(x_1, T(x_1)) & \geq P_1(x_1^*, T(x_1^*)) \\
\frac{\partial P_2}{\partial x_2} & \leq \frac{\partial P_2}{\partial x_2}.
\end{align*}
\]

For any strategy \(x_1\), the optimal reaction for the follower \(T(x_1)\) is obtained by solving

\[
\min_{x_2} P_2(x_1, x_2).
\]

The leader's optimal strategy is found by solving

\[
\min_{x_1} P_1(x_1, T(x_1))
\]

or equivalently

\[
\min_{x_1, x_2} P_1(x_1, x_2)
\]

s.t. \(P_2(x_1, x_2) = \min_y P_2(x_1, y)\).

In the above example, from (4) \(x_2 = T(x_1) = x_1/2\) so that
Game theory and transportation systems modelling

\[ P_i(x_1, T(x_1)) = x_1^2 + x_1 \]

and

\[ \frac{\partial P_i}{\partial x_1} = 2x_1 + 1 = 0. \]

Thus

\[ x_1^* = -\frac{1}{2}, \quad P_1 = -\frac{1}{4} \]  
\[ x_2^* = -\frac{1}{4} \quad P_2 = \frac{7}{16}. \] \(11\) \(12\)

For our purposes, the most interesting application of the Nash noncooperative game is in a situation where each player is trying to minimize his performance function without prior knowledge of the other player’s function. The game is a multi-period process in which, in a given period, each player’s strategy is based on the other player’s strategy in the previous period. Thus in period \(k\) the solution \((x_1^k, x_2^k)\) is found by solving

\[
\min_{x_1} P_i(x_1, x_2^{k-1})
\]

\[
\min_{x_2} P_2(x_1^k, x_2)
\]

and setting \(x_1^k = x_1, x_2^k = x_2\). Let \(F_i = (\partial P_i/\partial x_i)\), then the optimality conditions are

\[
F_i(x_1, x_2^{k-1}) = 0
\]

\[
F_2(x_1^k-1, x_2) = 0.
\] \(14\)

The iterative procedure defined by \(14\) is equivalent to the nonlinear block Jacobi procedure (Ortega and Rheinboldt, 1971) for solving the system of nonlinear equations

\[ F(x) = 0, \quad F = (F_1, F_2), \quad x = (x_1, x_2) \]

and the components of \(x_i\) constituting block \(i\). Convergence requirements for this procedure would indicate whether or not the process would converge to the Nash equilibrium solution as \(k \to \infty\). For example in the simple problem above

\[ F_1 = 2x_1 - x_2 + 1 \]

\[ F_2 = -x_1 + 2x_2 \] \(15\)

and the coefficient matrix

\[
A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}
\]

is strictly diagonally dominant; this is a sufficient condition for convergence of the sequence \((x_1^k, x_2^k)\) \(k = 1, 2, \ldots\) to the equilibrium solution from any initial starting point \((x_1^0, x_2^0)\) (Varga, 1962).

If player 1 knew the performance function of player 2, he would be in a position of choosing a Stackelberg strategy which would result in a lower value of \(P_1\) than that obtained in a Nash game. This is because at equilibrium the condition \(x_2 = T(x_1)\) is satisfied in both cases but in the Stackelberg game, \(x_1\) is chosen so that the function \(P_i(x_1, T(x_1))\) is minimized. If player 2 did not know \(P_i(x_1, x_2)\), his optimal recourse would be to assume the role of follower and adopt the strategy which solves \(8\). If he had knowledge of \(P_i(x_1, x_2)\), he might attempt to upset player one’s role as leader by choosing his own Stackelberg strategy or a short term strategy that would be particularly unfavorable to player 1.
EQUIVALENT STACKELBERG PROBLEMS

These arise in situations where a single supplier of transportation services or regulating agency wishes to determine an optimal operations plan, rates, and/or controls, taking into account user response. The supplier or regulator plays the leader role and users collectively play the role of follower. Let \( x_1 \in \mathcal{S}_1 \subseteq \mathbb{R}^n \) be the vector containing the leader decision variables, \( x_2 \in \mathcal{S}_2 \subseteq \mathbb{R}^n \) be the follower decision vector, and \( x_2 = T(x_1) \) is the user response for a given \( x_1 \), where \( T: \mathcal{S}_1 \subseteq \mathbb{R}^n \rightarrow \mathcal{S}_2 \). The leader's problem is to minimize the performance function \( P_1(x_1, T(x_1)) \) with respect to \( x_1 \), taking into account user response, i.e.

\[
\min_{x_1 \in \mathcal{S}_1} P_1(x_1, T(x_1)) \tag{16}
\]
or alternately

\[
\min_{x_1 \in \mathcal{S}_1, x_2 \in \mathcal{S}_2} P_1(x_1, x_2)
\text{ s.t. } x_2 = T(x_1). \tag{17}
\]

Usually it is not possible to obtain \( T(x_1) \) explicitly and the constraints (17) will take an alternative form which incorporates this relationship implicitly.

As an example, consider the signal optimization problem where the objective of the regulating agency is to minimize a network performance function \( P \) such as total travel time or gas consumption, with fixed origin/destination travel, but travellers choosing routes on the network in a user optimal fashion. \( \mathcal{S} \) is the set of feasible signal control variables, and \( \mathcal{F} \) the set of feasible link flow variables. For any given \( s \in \mathcal{S} \) there is a user optimal link flow solution \( f(s) \in \mathcal{F} \) and the regulator's problem (16) is to solve

\[
P_1: \quad \min_{s \in \mathcal{S}} P(s, f(s)). \tag{18}
\]

This corresponds to formulation (9) of the Stackelberg problem; since usually the functional form of \( f(s) \) is not known it is desirable to transform it into formulation (10) with user behavior included as constraints. This can be accomplished in a general manner by expressing user equilibrium conditions as a nonlinear complementarity problem (NCP) (Aashtiani, 1979). Let \( h \in \mathbb{R}^p \) be a path flow vector for the \( P \) network paths, and \( u \in \mathbb{R}^l \) be the O/D travel cost vector, where \( l \) is the number of O/D pairs. The NCP expressing the user optimality conditions for a given \( s \in \mathcal{S} \) is

\[
F(y, s) \cdot y = 0; \quad F(y, s), \quad y \in \mathbb{R}^{p+l}_+ \tag{19}
\]

where

\[
F = \begin{bmatrix} F_1^1 \\ F_2^1 \\ F_2^2 \end{bmatrix} \quad y = \begin{bmatrix} y_1^1 \\ y_2^1 \\ y_2^2 \end{bmatrix}
\]

\[
F_1^1 = C_s - u_i \quad r \in \mathcal{P}_i \quad i \in \mathcal{S} \tag{20}
\]

\[
F_2^1 = \sum_{r \in \mathcal{P}_i} h_r - d_i \quad i \in \mathcal{S}. \tag{21}
\]

\[
y_1^r = h_r \quad \text{all } r \in \mathcal{P} \tag{22}
\]

\[
y_2^i = u_i \quad \text{all } i \in \mathcal{S}. \tag{23}
\]

\( \mathcal{I} \) is the set of O/D pairs, \( \mathcal{P}_i \) is the set of paths between \( i \in \mathcal{S} \), \( \mathcal{P} = \bigcup_{i \in \mathcal{S}} \mathcal{P}_i \), \( C_s(h, s) \) is the travel cost on path \( r \in \mathcal{P} \), \( d_i \) is the number of trips between \( i \in \mathcal{S} \). Also let \( A \) be the
link path incidence matrix for the network, i.e.
\[ a_{i} = \begin{cases} 1 & \text{if link } l \text{ lies on path } r \\ 0 & \text{otherwise,} \end{cases} \] (24)

then \( h \) induces the link flow pattern \( f \) given by
\[ f = Ah. \] (25)

\( P1 \) is then equivalent to

\[ P2: \min_{s \in S, f} P(s, f) \]
\[ \text{s.t. } F(y, s) \cdot y = 0 \] (27)
\[ F(y, s) \geq 0 \] (28)
\[ y \geq 0 \] (29)

together with the definitional constraints (20)–(25).

This problem can be simplified in a variety of ways. From (27) \( u \) can be obtained as
\[ u_{i} = \frac{1}{d_{i}} \sum_{r \in \mathcal{R}_{i}} C_{r} h_{r} \quad \text{all } i \in \mathcal{I} \] (30)

reducing \( P2 \) to

\[ P3: \min_{s \in S, f} P(s, f) \]
\[ \text{s.t. } C \geq \frac{1}{d} \sum_{r \in \mathcal{R}_{i}} C_{r} h_{r} \quad \text{all } \mathcal{P} \] (31)

where \( \mathcal{H} \) is the set of feasible path flows. This is the formulation given in Tan et al. (1979).

Alternatively the NCP (27)–(29) can be replaced by an equivalent VI, resulting in

\[ P4: \min_{s \in S, f} P(s, f) \]
\[ \text{s.t. } c(f, s) \cdot (e - f) \geq 0 \quad \forall e \in \mathcal{E} \] (32)

where \( c(f, s): \mathcal{S} \subset R^{L} \rightarrow R^{L} \) is the link travel cost vector. This formulation is given in Marcotte (1981).

If for any given \( s \in \mathcal{S} \), \( c(f, s) \) is monotone, i.e.
\[ [c(f, s) - c(e, s)] \cdot [f - e] \geq 0 \quad \forall e, f, \in \mathcal{E} \]
then (32) is equivalent to the maxmin problem (Zukovitskii et al. (1973))

\[ \max_{f \in \mathcal{E}} \min_{e \in \mathcal{S}} c(f, s) \cdot (e - f). \]

Let \( w(f, s) = \min_{e \in \mathcal{E}} c(f, s) \cdot (e - f) \) then \( P4 \) is equivalent to
Finally if the Jacobian matrix $\frac{\partial c}{\partial f}$ is symmetric, the conditions (27)-(29) can be derived as the Kuhn–Tucker conditions of the optimization problem (Beckmann et al., 1956)

$$\min_{f \in \mathcal{F}} z(f, s) \quad \text{where} \quad z(f, s) = \sum_{e \in \mathcal{E}} c(x, s) \, dx$$

and $\mathcal{F}$ is the set of network links.

$P1$ is then equivalent to

$$\min_{f \in \mathcal{F}} P(s, f)$$

s.t. $z(f, s) = \min_{e \in \mathcal{E}} z(e, s)$.

In transportation research, formulations $P1$, $P3$ and $P4$ have formed the basis for an investigation of solution algorithms (see Abdulal and LeBlanc (1979), Tan et al. (1979), Marcotte (1981)), however to date, no convergent algorithm has been found which would effectively solve large scale problems.

$P4$ fits into a general class of infinitely constrained problems known as the Fritz John problem (1948) and its properties and solution algorithms have been studied with regard to other game theory applications (e.g. Blankenship and Falk, 1976). Optimization problems with optimization problems as constraints, such as $P6$, have been studied and applied to hierarchical resource allocation problems (e.g. Bard 1983, Shimiza and Aiyoshi, 1981).

In the most general problem, the signal variables which can be set by the controlling agent include green and cycle times, and offsets. By specifying the cost functions $c_i$ for each network link in terms of these variables, the performance function $P$ can be constructed as

$$P(s, f) = \sum_{l \in \mathcal{L}} f(c_l(f, s)).$$

The set $\mathcal{L}$ is defined by linear constraints which: relate green times on different intersection approaches to the cycle time, introduce possible upper and lower bounds for the green and cycle times, provide loop constraints which ensure that the sum of the offsets around any loop of the network is an integral multiple of the cycle time. (This introduces an integer control variable into the model.) Gartner et al. (1975) have formulated a signal optimization model of this form but they do not include a user response model, link flows being assumed given as input. If the route choice model is added as a constraint, a Stackelberg problem of the form presented above will result.

As an example, consider the simple network model shown in Fig. 1. The set $\mathcal{L}$ of O/D pairs is \{(a, b), (c, d)\} and there is a signal at the intersection of links 1 and 3. The cost functions used are

$$c_1 = \frac{f_1}{g_1} + 2 \quad c_2 = 2f_2 \quad c_3 = \frac{2f_3}{g_3}$$

where $g_l$ is the green time on link $l$. Also

$$g_1 + g_3 = 20, \quad d_1 = d_2 = 10.$$
For this problem it is straightforward to obtain formulation $P_1$ using the user optimal conditions between O/D pair 1 to eliminate the flow variables. Solving the equations

\[
\frac{f_1}{g_1} + 2 = 2f_2 \\
f_1 + f_2 = 10
\]

gives

\[
f_1 = \frac{18g_1}{1 + 2g_1}.
\]

Using $g_1 - 20 - g_1$ the performance function is

\[
P = 10c_1 + 10c_3 = \frac{200 + 40g_1}{1 + 2g_1} + \frac{200}{20 - g_1}.
\]

Solving $\frac{\partial P}{\partial g_1} = 0$ for $g_1$ leads to the solution:

\[
g_1 = 7.7306, \quad g_3 = 12.2694, \quad f_1 = 8.4533, \quad f_2 = 1.5467.
\]

This problem could also be solved using any of the other formulations where instead of eliminating the flow variables, the route choice conditions are included as constraints. A possible approach based on $P_5$ is now described and applied to this example.

A PENALTY APPROACH TO SOLVING STACKELBERG PROBLEMS

In this section we introduce a new possible solution approach suggested by formulation $P_5$ of the signal optimization problem. Note that $w(f, s)$ has the following properties:

(i) $w(f, s)$ is continuous (assuming $c(f, s)$ is continuous).

(ii) The inequality $w(f, s) \leq 0$ is satisfied for any $f \in \mathcal{F}$, $s \in \mathcal{S}$.

(iii) $w(f, s) = 0$ if and only if $f$ is the user optimal solution corresponding to a given signal control vector $s \in \mathcal{S}$ (see theorem 5 in Zikhovitshii et al., 1973).

Then $w(f, s)$ has the properties required for a penalty function so that an approximate solution to $P_5$ could be obtained by solving

\[
P7: \max_{s \in \mathcal{S}} \max_{f \in \mathcal{F}} \left\{ -P(f, s) + \mu w(f, s) \right\} \tag{33}
\]
where $\mu$ is chosen large enough to ensure that $w(f^*, s^*) \sim 0$. Properties and solution algorithms for this problem are presently being investigated. It is likely that methods being developed for solving the user optimal route choice problem by maximizing the "gap" function $w$ could be applied here. Also under certain conditions solving (33) is equivalent to determining the saddle point of $\phi(x, e)$ given as

$$
\phi(x, e) = -P(f, s) + \mu c(f, s) \cdot (e - f)
$$

(34)

with $x = (f, s)$ (see Dem'yanov and Malozemov, 1974). Algorithms for this problem have been developed in the context of game theory (see Ermolev (1966) for a review).

If the example above is solved using the penalty formulation $P7$, we have to solve

$$
\max_{\{f_i, x_i\}} \min \left\{ -[f_i^2/g_i + 2f_i + 2(10 - f_i)^2 + 200/(20 - g_i)] \right\}
$$

$$
+ \mu(f_i/g_i + 2f_i - 18)(e_i - f_i)
$$

where the feasibility constraints have been used to eliminate $f_2, g_3$ and $e_2$. This problem was solved numerically for different values of $\mu$; the results are given in Table 1. When $\mu = 0$ the route choice solution corresponds to the system optimal solution. However with $\mu = 1$ it can be seen that the solution already corresponds closely to the exact solution given above.

In many other Stackelberg type problems, the user (follower) conditions can be expressed as a VI or NCP (see Fisk and Boyce, 1983a), and formulations similar to those presented above can be obtained. There are also other problems which resemble Stackelberg problems; these include the maximum likelihood or least squares model calibration problem (Boyce et al., 1983) and the trip matrix estimation problem, (Nguyen, 1981) where an entropy function is maximized with a travel choice model as a constraint.

**INNER-OUTER ITERATIVE SCHEMES FOR STACKELBERG PROBLEMS**

Solution of transportation problems which fit into the Stackelberg framework have usually been obtained using heuristics because known rigorous approaches have been found to be inefficient for large scale problems. A heuristic which is frequently applied is to sequentially solve the leader and follower problems (e.g. Tan et al., 1979); in iteration $k + 1$ the problem

$$
\min_{x_i \in \mathcal{X}_i} P_i(x_i, x_{-i}^k)
$$

**Table 1**

<table>
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<th>$\mu$</th>
<th>$f_1$</th>
<th>$g_1$</th>
<th>$P$</th>
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is solved, and $x_2^{k+1}$ is then obtained as

$$x_2^{k+1} = T(x_1^{k+1}).$$

Intuitively it might be expected that if the first player's strategy did not greatly influence the second player's optimal solution and vice-versa, this approach would converge to the exact solution. However this is not the case as is shown below for a general problem. First, as a special case, consider the simple two player problem formulated above, which is to solve

$$\min_{x_1,x_2} (x_1^2 - x_1x_2 + 2x_2^2 + x_1)$$

s.t. $x_4 = \frac{x_1}{2}$. \hspace{1cm} (35)

If $x_1^k = -1/2$, $x_2^k = -1/4$, which is the exact solution, $x_1^{k+1}$ is the solution to

$$\min_{x_1} P_1(x_1, -1/4) = \min_{x_1} (x_1^2 + x_1/4 + x_1)$$

giving $x_1^{k+1} = -5/8$, $x_2^{k+1} = -5/16$ and $P_1^{k+1} = -25/64$. This objective value is lower than $P_1(-1/2, -1/4)$ because (36) is solved without constraint (35).

Generally, this result can be expressed as follows. Let $G$ represent the mapping which takes $(x_1^k, x_2^k)$ into $(x_1^{k+1}, x_2^{k+1})$ so that with $y = (x_1, x_2)$, $y^{k+1} = G(y^k)$. If $y^k = y^*$, $y^{k+1} \neq G(y^*)$ because the problem

$$\min_{x_1 \in S_1} P_1(x_1, x_2^*)$$

can produce a value of $P_1$ smaller than $P_1(x_1^*, x_2^*)$. This is due to the fact that (37) is not constrained by the relationship

$$x_2^k = T(x_1^{k+1}).$$

This iterative scheme has the form of the successive approximation algorithm for solving the fixed point problem $y = G(y)$ (Ortega and Rheinboldt, 1970). However the above discussion shows that $y^*$ is not the fixed point of $G$ so that the sequence of points generated will not converge to $y^*$. If $G$ is a contraction mapping, it may converge to some point other than $y^*$. In fact the inner-outer approach corresponds to solving the Nash problem using the Gauss–Seidel approach and it can easily be shown that the sequence of points generated above converges to the Nash solution given by (11) and (12).

**Applications of Nash Noncooperative Games**

In addition to the well known user optimal route choice problem, this game model represents the equilibrium conditions for an oligopolistic market situation where $J$ firms (carriers) are competing for customers (shippers, passengers), and each firm seeks a production strategy (operations plan, rates) which would minimize its performance function (e.g. costs—revenue). A firm does not know the performance function of the other firms and the transition to equilibrium is a multi-period process in which any one firm’s strategy in a given period is a reaction to other firms’ decisions in the previous period (Friedman, 1977).

Let $y_j \in R^{n_j}$ be the vector of decision variables for carrier $j$, $y = (y_1, \ldots, y_J)$, $P_j$ is the performance function for $j$, and $x = T(y)$ represents the user response to $y$, then the equilibrium solution is obtained by simultaneously solving the $J$ Stackelberg problems:

\[\text{The author is grateful to Patrice Marcotte for bringing up this point.}\]
$P8: \min_{y \in \mathcal{Y}, x \in \mathcal{X}} P_j(x, y) \quad j = 1, \ldots, J \tag{38}$

subject to $x = T(y) \tag{39}$

$S_j \subset \mathbb{R}^n$ is the set of feasible decisions for firm $j$, and $\mathcal{X}$ is the set of feasible user responses.

The problem could be solved using a nonlinear block Jacobi procedure; in iteration $k + 1$ the $J$ problems

$$\min_{y \in \mathcal{Y}, x \in \mathcal{X}} P_j^k(x, y) \quad j = 1, \ldots, J$$

are solved, with

$$P_j^k(x, y) = P_j(x, y_1^k, \ldots, y_{j-1}^k, y_j, y_{j+1}^k, \ldots, y_J^k)$$

$$T_j^k(y_j) = T(y_1^k, \ldots, y_{j-1}^k, y_j, y_{j+1}^k, \ldots, y_J^k)$$

and $y_{j+1}^k$ is set equal to the solution $y_j$ to this problem.

Alternatively, a block Gauss–Seidel procedure could be used in which the $J$ problems in iteration $k + 1$ are solved sequentially, the solution for subproblem $j$ being used in the subproblem $j + 1$. As shown in Fisk and Boyce (1983a), the problem of simultaneously solving $(P8)$ for all firms can be formulated as a $VI$ and convergence properties of the above algorithms can be explored within this framework.

There are many alternative formulations for the competing firm problem. For example if one firm knows the performance function of the other firms, it may assume the role of leader with the other firms being followers but competing among themselves in a non-cooperative manner. (This problem is studied in Sherali et al., 1983.) Or there may be collusion between some of the firms in which case a composite performance function of the colluding firms would be minimized.

CARRIER COMPETITION FOR INTERCITY PASSENGER TRAVEL

As an example of a Nash problem, a model representing competition between carriers for intercity passenger travel is formulated. It could apply at a regional, national, or international level. Basically, the performance function specified for each carrier represents the net operating cost, with decision variables being vehicle frequencies on routes, and fares. Passengers’ choice of mode is obtained from a mode choice model, and route choice for a given mode is user optimal. In this example it is not possible to transfer between modes.

$\mathcal{M}$ is the set of modes serving the passengers (e.g. air, rail, bus, automobile), $\mathcal{J}$ is the set of carriers, and $\mathcal{M}_j$ is the set of modes operated by carrier $j$. For a given mode $m \in \mathcal{M}_j$, carrier $j$ operates vehicles on a set of routes $\mathcal{R}_jm$. A route has an origin and destination terminus between which vehicles are dispatched with a frequency $v_{jm}$. The intermediate nodes on a route are cities where pick-ups and drop-offs may occur. An adjacent node pair on a route constitutes a link, $\mathcal{L}_{jmr}$ is the set of links on $r$, and

$$\mathcal{L}_{jm} = \bigcup_{r \in \mathcal{R}_jm} \mathcal{L}_{jmr}$$

is the set of links in the mode $m$ network of carrier $j$.

The carrier decision variables are the frequency $v_{jm}$, for each route $r$ and mode $m$, and fares. It is assumed here that fares are determined on a link basis, $q_{ilm}$ being the passenger fare on link $l$ for mode $m$. Other fare policies are possible including different fare structures for each carrier. The structures used in a particular application would reflect the actual policies being used.
The carrier’s objective is to determine the elements† of \( v_j \) and \( q_j \) so as to minimize the net operating cost, given here as the total operating cost \( T_j \) minus the total passenger revenue \( R_j \). If \( f_{jm} \) is the passenger flow on \( l \in \mathcal{L}_{jm} \) the total revenue is

\[
R_j = \sum_{m \in \mathcal{M}_j} \sum_{l \in \mathcal{L}_{jm}} q_{jm} f_{jm}.
\]

Let \( G_{jm} \) designate the operating cost per vehicle for mode \( m \) on route \( r \). It is comprised of manpower, fuel, maintenance, and vehicle ownership costs. Some of these costs depend directly on travel time which in turn may depend on \( v_{jm} \) and \( f_{jm} \) through congestion effects. (The costs will also depend on other factors which have been assumed fixed in this problem.) This dependency is recognized by writing \( G_{jm} \) as \( G_{jm}(v_{jm}, f_{jm}) \); its functional form could be derived mathematically using probability and queueing theory. The total operating cost for carrier \( j \) is

\[
T_j = \sum_{m \in \mathcal{M}_j} \sum_{l \in \mathcal{L}_{jm}} v_{jm} G_{jm}(v_{jm}, f_{jm}).
\]

The set of feasible decisions \( \mathcal{S}_j \) for carrier \( j \) are defined by the flow conservation constraints at termini:

\[
\sum_{r \in \mathcal{S}_n} v_{jm} = \sum_{r \in \mathcal{S}_n} v_{jm}, \quad \forall n \in \mathcal{N}_{jm}, \quad m \in \mathcal{M}_j
\]

(40)

where \( \mathcal{N}_{jm} \) is the set of termini for carrier \( j \) mode \( m \), \( \mathcal{S}_n \) is the set of routes leaving \( n \), \( \mathcal{S}_n \) is the set of routes entering \( n \), and non-negativity conditions

\[
v_{jm} \geq 0, \quad q_{jm} \geq 0 \quad \forall m \in \mathcal{M}_j, \quad r, l.
\]

(41)

The passenger network is a multimodal network, the set of links for each mode being the union of all carrier links for that mode, plus intra and inter carrier transfer links. If the automobile is a possible mode, then it is also necessary to add the road network. Thus the total set of passenger links is

\[
\mathcal{L} = \bigcup_{m \in \mathcal{M}} \mathcal{L}_{jm} + \bigcup_{m \in \mathcal{M}} \mathcal{L}_{im} + \mathcal{L}_a
\]

where \( \mathcal{L}_{jm} \) is the set of transfer links for mode \( m \), and \( \mathcal{L}_a \) is the set of road links. The set \( \mathcal{M}' \) is defined to be the union of \( \mathcal{M} \) and the auto mode.

Let \( \mathcal{S} \) be the set of city pairs, then for a given mode \( m \in \mathcal{M}' \) passengers are assumed to choose routes in a user optimal fashion. This can be expressed mathematically as

\[
C_{mk}(q, v, h) - u_{mi} = 0 \quad \text{if} \quad h_{mk} > 0
\]

(42)

\[
C_{mk}(q, v, h) - u_{mi} \geq 0 \quad \text{if} \quad h_{mk} = 0 \quad k \in \mathcal{P}_{mi}, \quad i \in \mathcal{S}
\]

\[
\sum_{k \in \mathcal{P}_{mi}} h_{mk} = d_{mi} \quad \forall i \in \mathcal{S}
\]

(43)

\[
h_{mk} \geq 0 \quad \forall k \in \mathcal{P}_{mi}, \quad i \in \mathcal{S}
\]

(44)

where \( \mathcal{P}_{mi} \) is the total set of paths in the passenger network between \( i \in \mathcal{S} \), and by mode \( m \), \( h_{mk} \) is the passenger flow on \( k \in \mathcal{P}_{mi} \), \( u_{mi} \) is the equilibrium travel cost between \( i \) by mode \( m \), \( C_{mk}(q_{mi}, v_m, h_m) \) is the passenger travel cost on path \( k \in \mathcal{P}_{mi} \), denoted here for generality, as a function of \( q_m \), \( v_m \) and \( h_m \).

†We use the convention that if a variable \( x_{ik} \) has been introduced, \( x_q \) is the vector with elements \( x_{ik} \) and \( x_i \) is a matrix with elements \( x_{ik} \).
$d_{mi}$ is the demand for travel by mode $m$ between $i$ obtained from the travel demand function $H_m(u)$; i.e.

$$d_{mi} = H_m(u) \quad \text{all } i \in \mathcal{I}, \ m \in \mathcal{M}.$$  \hspace{1cm} (45)

For example, if a logit model is used, $H_m(u)$ may have the form

$$H_m(u) = \frac{d_i e^{-\theta u_i}}{\sum_{m \in \mathcal{M}} e^{-\theta u_i}} \quad \text{all } i, m$$

where $d_i$ is the total demand for travel between $i$ (assumed given), and $\theta$ is the dispersion parameter.

To complete the model, a definitional constraint relating $f$ to $h$ is required. We have

$$f_{jmi} = \sum_{k \in \mathcal{P}_{mi}} a_{ik} h_{mk} \quad \text{all } i, m, j$$  \hspace{1cm} (46)

where

$$a_{ik} = 1 \text{ if } l \in \mathcal{L}_{pi} \text{ lies on path } k \in \mathcal{P}_{pi} \text{ otherwise}$$

$P8$ then has the form

$$\min_{u_{h_{ik}}, y_{j \mathcal{P}_{mi}}} \quad T_j - R_j$$

s.t. (42)--(45) simultaneously for all $j \in \mathcal{J}$

and the definitional constraint (46). The user behavior conditions can be expressed as an NCP (Aashtiani, 1979), or a VI (Fisk and Boyce, 1983b) and other formulations can be derived from these, as was done for the signal optimization problem. Solving (47) using the block Gauss-Seidel procedure outlined above involves in one iteration solving each of the $J$ Stackelberg problems in (47) independently, the decision variables of the other carriers being held fixed at their values obtained from the previous solution.

The above model will produce the Nash equilibrium solution to the competing carrier problem taking into account passenger behavior. It would be a suitable tool for determining impacts of policy or regulatory decisions on this segment of the transportation industry particularly at the national level. At the individual carrier level, if one carrier knew the performance functions of the other carriers, it would be in the position of being able to play a leader role as in the model of Sherali et al. (1983), and a different equilibrium would result. Otherwise an optimal operations plan and fare structure are likely to be determined in a given time period by solving a single Stackelberg problem, the variables of the other carriers being fixed at their observed values from the previous time period. As discussed above, if the interactions between the carriers are weak, this process will converge over time to the Nash equilibrium solution (given that other factors in the transportation environment do not change).

PROBLEMS WITH HIGHER HIERARCHICAL LEVELS OF DECISION-MAKERS

Finally we note that the problem of determining optimal regulatory policies for competing carriers has an equivalent Stackelberg formulation. Suppose the regulatory agency wishes to determine a policy which optimizes a performance function $P(y, z)$ where $z$ is the vector of decision variables for the agency, and $y$ is the composite decision vector of the carriers, then the agency's problem is:
\[
\min_{y, z} P(y, z) \quad \text{s.t. } y = R(z).
\]

\(R(z)\) is the carriers' response to \(z\), implicitly given as the solution to the combined carriers' problem \((P8)\) for a fixed value of \(z\).

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