

ON THE STABILITY OF USER EQUILIBRIA IN STATIC TRANSPORTATION NETWORKS

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The stability of a user equilibrium determines whether it could be realized or sustained in a transportation network, when subject to perturbations in traffic conditions and drivers' route choice behaviors. In this paper, with a simple deterministic dynamical system model of the traffic assignment problem, we first develop a systematic approach to analyzing asymptotic stability and instability of user equilibria in static transportation networks with fixed demand. We then discuss two examples of non-monotone traffic assignment problem. Compared with existing approaches, the new approach is simpler to use and leads to results consistent with the former. This study could be helpful for analysis and design of real transportation networks.

KEYWORDS: Static user equilibrium, asymptotic stability, instability, non-monotone traffic assignment problem

1. INTRODUCTION

Traffic assignment problem, one of the core steps in transportation planning, is to assign origin-destination traffic demand to individual paths and links of a road network. Such a traffic assignment process is usually based on the observation of the existence of user equilibrium (UE) on average. According to Wardrop's first principle (Wardrop, 1952), at UE, "the journey times of all routes actually used are equal and less than those which would be experienced by a single vehicle on any unused route". In static transportation networks, a link performance function is assumed between the travel time of a link and traffic volumes on this and other links. For continuous link performance functions, there exists at least one user equilibrium assignment (Nagurney, 1993). A widely used method for solving UE in road networks is the mathematical programming formulation (Beckmann et al., 1956), which was also used by Samuelson (1952) to solve spatial price equilibrium problems. This formulation can only be applied when all link performance functions are separable or symmetric (Dafermos and Sparrow, 1969; Dafermos, 1971) and has been extended for multi-class problems (Dafermos, 1972) and for problems with elastic demands (Dafermos, 1982a). In these cases, UE usually is unique and stable for increasing link performance functions (Sheffi, 1984). When link performance functions are asymmetric, the mathematical programming formulation is no longer applicable, and the variational inequalities formulation can be used (Smith, 1979; Dafermos, 1980; Hearn, 1982). It has been proved that the solution is unique and stable for monotone link performance functions, which corresponds to a positive Jacobian matrix of path travel times with respect to path volumes (Smith, 1979; Dafermos, 1980).

In real-world transportation networks, however, link performance functions could be non-monotone due to complicated interactions between different links, caused by merging/diverging traffic, queue-spillbacks, and signalized or non-signalized intersections. Some examples of such situations were given in Watling (1996). For non-monotone traffic assignment problems, there could exist multiple or unstable UE. For examples, in Netter (1972), a network of two classes of users can have three UE, which can be stable or unstable; in Yang (2005, Chapter 5), a single unstable UE was reported.

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It is important to analyze the stability property of a UE, since it is highly related to whether the UE could be realized or sustained when subject to perturbations in traffic conditions and drivers' route choice behaviors. The stability of a UE is also related to whether a numerical method can successfully find it. Therefore stability analysis of UE can have significant implications on both analysis and design of transportation networks.

In literature, there have been a number of approaches to defining and analyzing stability of UE. In the seminal work of Smith (1979) and Smith (1984b), the stability of a UE was defined for a deterministic traffic shifting process. In Watling (1996) and Watling and Hazelton (2003), a UE is said to be (locally) stable "unless arbitrarily small deviations may cause the system to diverge from the original equilibrium state". In Horowitz (1984), an equilibrium is defined to be stable if "it is unique and the link volumes converge over time to their equilibrium values regardless of the initial conditions." In Nagurney and Zhang (1996) and Zhang and Nagurney (1996), systematic definitions and discussions of the stability of a user equilibrium was given with a projected dynamical system. Existing studies yield quite consistent results on stability: for symmetric or monotone traffic assignment problems, the unique user equilibrium is always stable (Smith, 1979, 1984b; Nagurney and Zhang, 1996; Zhang and Nagurney, 1996; Zhang et al., 2001); for non-monotone problems, there could exist multiple or unstable user equilibria (Netter, 1972; Watling, 1996; Marcotte and Wynter, 2004; Yang, 2005).

Among the aforementioned definitions, those based on dynamical processes can be systematically applied to analyze stability of UE. However, existing dynamical processes are usually very complicated in form. In this paper, based on a deterministic dynamical system model of the traffic assignment problem recently developed in Jin (2007), we propose a new systematic approach to analyzing asymptotic stability of UE in static transportation networks, especially non-monotone ones. Here the discussions of the stability of UE are rigorous and consistent with those in the mathematical literature of dynamical systems (Birkhoff, 1927). This new dynamical system is simpler in form and can simplify the analysis of stability of UE. In addition, since the new dynamical system formulation is still well defined when traffic demand is elastic and traffic dynamics are considered, the new approach could be well extended for analyzing the stability of UE in more general transportation networks.

The rest of the paper is organized as follows. In Section 2, we review existing deterministic dynamical systems of the static traffic assignment problem with fixed demand, especially the dynamical systems in Smith (1984b) and Jin (2007) and present a definition of stability. In Section 3, we discuss the properties of the dynamical system in Jin (2007) and study stability of for special types of UE. In Section 4, we discuss two examples of non-monotone traffic assignment problem. We make concluding remarks in Section 5.

2. DYNAMICAL SYSTEM MODELS OF THE TRAFFIC ASSIGNMENT PROBLEM AND STABILITY OF UE

In this study, we denote an origin-destination (O-D) pair by $r-s$ and a path by k . Further, between O-D pair $r-s$, q_{rs} is traffic demand, f_k^{rs} the assigned flow on path k , c_k^{rs} the resultant travel time on path k , and v_{rs} the average travel time. Then we have the following basic relationships:

$$q_{rs} = \sum_k f_k^{rs}, \forall r, s, \quad (1)$$

$$f_k^{rs} \geq 0, \forall k, r, s. \quad (2)$$

Here we assume the dimension of rs -subspace as K_{rs} , which is the number of all possible used paths connecting O-D pair $r-s$. In a general network, K_{rs} could be very large theoretically, but relatively small practically, since most of the paths connecting O-D pair $r-s$ are much longer than others and never used by any vehicles. If we denote \mathbf{P}_k^{rs} as the unit vector with the (rs, k) th element as 1 in a real finite-dimensional vector space $\mathfrak{R}^{\Pi_{rs} K_{rs}}$, \mathbf{P}_k^{rs} for the same O-D pair $r-s$ forms the basis for rs -subspace. For any assignment vector \mathbf{f} of dimension $\Pi_{rs} K_{rs}$, we write it in the following form

$$\mathbf{f} = \sum_{rs} \mathbf{f}_{rs} = \sum_{rs} \sum_k f_k^{rs} \mathbf{P}_k^{rs}.$$

Here $\mathbf{f}_{rs} = \sum_k f_k^{rs} \mathbf{P}_k^{rs}$ is the corresponding assignment vector in the rs -subspace. We denote the set of all \mathbf{f}_{rs} by F_{rs} , and the set of all \mathbf{f} by F . We can see that F_{rs} are mutually exclusive with respect to O-D pairs, and the union of F_{rs} yields F . Further, F_{rs} defined by equations (1) and (2) is the feasible set of traffic assignment problem in rs -subspace, and F is the feasible set in the concerned space. From equations (1) and (2), F_{rs} is a convex hull with $q_{rs} \mathbf{P}_k^{rs}$ ($k=1, \dots, K_{rs}$) as its vertices and generally not a subspace, since vector 0 cannot be a feasible assignment when $q_{rs} > 0$.

Here c_k^{rs} is a function of \mathbf{f} , which can be obtained by summing all related link performance functions of \mathbf{f} . Then $\forall m, n, l, r, s, k$, a static traffic assignment problem is monotone when $\sum_{rs} \sum_k (c_k^{rs}(\mathbf{f}) - c_k^{rs}(\mathbf{g})) (f_k^{rs} - g_k^{rs}) > 0$ for any assignments \mathbf{f} and \mathbf{g} . It is equivalent to saying that, a traffic assignment problem is monotone, if and only if the corresponding Jacobian matrix of path travel times with respect to path flows $(\partial c_k^{rs} / \partial f_l^{mn})$ is positive definite (Dafermos, 1980).

2.1 Dynamical system models

As economic equilibria (Walras, 1954; Marshall, 1961), UE in transportation networks can also be considered as the result of a day-to-day tatonnement or adjustment process of drivers' route choice behavior. Such behaviors can be considered as stochastic route choices of individual drivers at the disaggregate level (Horowitz, 1984) or deterministic traffic shifting process at the aggregate level (Smith, 1979; Friesz et al., 1994; Nagurney and Zhang, 1996; Jin, 2007). Here we only review two relatively simpler deterministic dynamical systems by Smith (1984b) and Jin (2007). These and other deterministic dynamical systems describe the day-to-day changing trajectories of path flows and are path-based formulations (Jayakrishnan et al., 1994). They can be written as the following system of ordinary differential equations

$$\dot{f}_k^{rs} = R_k^{rs}(\mathbf{f}), \quad (3)$$

where $R_k^{rs}(\mathbf{f})$ is the rate of change in the flow on path (rs, k) . Here $\dot{f}_k^{rs} = \frac{d}{d\tau} f_k^{rs}$, and τ is the time variable for route choices. For fixed demand problems, we have $\sum_k R_k^{rs}(\mathbf{f}) = 0$.

For an O-D pair $r-s$, if an assignment \mathbf{f} satisfies $\dot{f}_k^{rs} = 0$ for all k , then it is called an equilibrium in the rs -subspace. We can see that \mathbf{f} is an equilibrium in the rs -subspace, if and only if $R_k^{rs}(\mathbf{f}) = 0$ for any $k = 1, \dots, K_{rs}$. In addition, \mathbf{f} is an equilibrium of dynamical system (3), if and only if it is an equilibrium in all rs -subspaces.

2.1.1 Smith's dynamical system

In Smith's dynamical system (Smith, 1984b), which is the first dynamical system model of the traffic assignment problem, the rate of change in a path flow can be written as follows

$$R_k^{rs}(\mathbf{f}) = -\sum_j f_{k \rightarrow j}^{rs} + \sum_j f_{j \rightarrow k}^{rs}, \quad (4)$$

where the flow swapped from path k to j is

$$f_{k \rightarrow j}^{rs} \equiv f_k^{rs} (c_k^{rs} - c_j^{rs})_+, \quad (5)$$

and the flow swapped from path j to k is

$$f_{j \rightarrow k}^{rs} \equiv f_j^{rs} (c_j - c_k)_+, \quad (6)$$

with a function $(x)_+ \equiv \max\{x, 0\}$. The principle for changing path flows in equation (4) is that flows are swapped from paths with larger travel times to those with shorter, and an equilibrium of equation (4) is always a UE. By constructing a Lyapunov function (Liapunov, 1966), Smith (1984b) proved the stability of the unique user equilibrium for monotone path travel time functions. In Peeta and Yang (2003), it was shown that the objective function of the mathematical programming formulation (Beckmann et al., 1956) can be also considered as a Lyapunov function of equation (4). The latter proof, however, only applies for symmetric link performance functions.

2.1.2 The FIFO dynamical system

In Jin (2007), a new dynamical system was proposed with a new rate of change in a path flow

$$-R_k^{rs}(\mathbf{f}) = J_k^{rs} \equiv q_{rs} f_k^{rs} (c_k^{rs} - v_{rs}), \quad \forall k, r, s, \quad (7)$$

with the average travel time of all vehicles from origin r to destination s

$$v_{rs} = \frac{\sum_j c_j^{rs} f_j^{rs}}{q_{rs}}, \quad \forall r, s. \quad (8)$$

Here J_k^{rs} is the First-In-First-Out (FIFO) violation among path flows, since they are zero if and only if all used paths of the same O-D pair share the same travel times; i.e., all vehicles of the same O-D pair observe FIFO principle. Note that, in literature, FIFO is often used to refer to the queuing principle on a roadway: if vehicles observe FIFO principle, vehicles entering a roadway earlier leave it earlier. That is, with FIFO

principle, the orders of vehicles at the entrance and the exit of the roadway are the same. Similarly, if we compare the orders of vehicles at an origin and the corresponding destination, we can also determine whether they follow FIFO principle. In static traffic assignment problem, there is no FIFO issue on a link, or FIFO principle is always observed. However, FIFO principle could be violated due to different travel times on different paths. We hereafter call equation (7) the FIFO dynamical system.

Actually, equation (7) can be also be rewritten in the form of equation (4) but with

$$f_{k \rightarrow j}^{rs} = -f_{j \rightarrow k}^{rs} = \frac{1}{2} f_k^{rs} f_j^{rs} (c_k^{rs} - c_j^{rs}), \quad (9)$$

which could be positive or negative. The principle for changing path flows in the FIFO dynamical system is to reduce flows from paths with higher travel times than the average O-D travel time. Compared with other dynamical systems, the FIFO dynamical system is simpler in form without using piece-wise rates of change in path flows. Another distinctive property of the FIFO dynamical system is that its equilibrium may not be a UE, although a UE is always its equilibrium. That is, in an equilibrium of equation (7), there may exist unused shorter paths. Such an equilibrium is called a partial user equilibrium (PUE), which can be considered as a UE of a path set excluding those unused shorter paths. However, PUE are always unstable for the FIFO dynamical system. That is, stable equilibria are always UE. Further, the FIFO dynamical system is stable at UE for symmetric link performance functions, with the objective function by Beckmann et al. (1956) being its Lyapunov function (Jin, 2007).

2.2 Definition of stability

Here we define the stability of UE with respect to the FIFO dynamical system (7). That is, we call a UE stable if and only if it is a locally asymptotically stable equilibrium for the FIFO dynamical system. Therefore, an equilibrium is stable if and only if, starting from any non-equilibrium initial state close to it, the solution trajectory will converge to it for $\tau \rightarrow \infty$.

Then, we apply the following general stability and instability theorem to determine stability of UE (LaSalle, 1960; Liapunov, 1966; Strogatz, 1994; Hairer et al., 1993).

THEOREM 1 Sufficient conditions of stable and unstable equilibrium: *An equilibrium is stable, if all eigenvalues of the linearized system have negative real parts; it is unstable, if one eigenvalue has positive real part.*

Similarly, we can also define stability of UE with respect to Smith's dynamical system (4) and have corresponding sufficient conditions of stable and unstable UE. In this study we will focus on analyzing stability of UE with the FIFO dynamical system. We can see that, \mathbf{f} is a stable equilibrium of equation (7), if and only if it is a stable equilibrium in all rs -subspaces. Therefore, in the following, we only need to analyze the stability of equilibrium in rs -subspace.

3. ANALYSIS OF STABILITY OF UE

Before discussing stability of UE, we start with more properties of the FIFO dynamical system.

3.1 Properties of the FIFO dynamical system

In rs -subspace, each vertex corresponds to an assignment that only one path is used by all vehicles of O-D pair $r-s$. For example, k th vertex, $q_{rs}\mathbf{P}_k^{rs}$, in the rs -subspace corresponds to an assignment. From equation (7), we can see that any vertex in rs -subspace is a trivial equilibrium in this subspace. That is, we have the following trivial equilibrium in rs -subspace: $\mathbf{f}_{rs} = \sum_i \bar{f}_i^{rs} \mathbf{P}_i^{rs} = q_{rs} \mathbf{P}_k^{rs}$, where $\bar{f}_k^{rs} = q_{rs}$ and $\bar{f}_j^{rs} = 0$ ($j \neq k, \forall r, s$). We hereafter call such an equilibrium as a vertex equilibrium.

THEOREM 2: *Any convex hull, whose vertices are a subset of vertices of F_{rs} , is an invariant set of the FIFO dynamical system (7).*

Proof: For a convex hull, \bar{F}_{rs} , whose vertices are a subset of vertices of F_{rs} , we can write its element \mathbf{f}_{rs} as $\mathbf{f}_{rs} = \sum_{k=1}^{\bar{K}_{rs}} \bar{f}_k^{rs} \mathbf{P}_k^{rs}$. Here $\bar{K}_{rs} \leq K_{rs}$, $\bar{f}_k^{rs} \geq 0$ ($k = 1, \dots, \bar{K}_{rs}$), and $\sum_{k=1}^{\bar{K}_{rs}} \bar{f}_k^{rs} = q_{rs}$. In the following, we show that, $\mathbf{f}_{rs}(\tau > 0) \in \bar{F}_{rs}$, if $\mathbf{f}_{rs}(\tau = 0) \in \bar{F}_{rs}$. First, for any $\mathbf{f}(\tau) \in F$, link cost c_k^{rs} and, therefore, $J_k^{rs}(\mathbf{f})$ are always finite. Therefore the solution trajectory of $f_k^{rs}(\tau)$ is continuous. Second, if $f_k^{rs}(\tau_0) = 0$ at any τ_0 , then $f_k^{rs}(\tau) = 0$ for any $\tau > \tau_0$, since $J_k^{rs}(\tau_0) = 0$. That is, the solution trajectory cannot cross $f_k^{rs} = 0$. Thus $f_k^{rs}(\tau) \geq 0$ for any τ , if $f_k^{rs}(\tau = 0) \geq 0$ for $k = 1, \dots, \bar{K}_{rs}$. Third, due to the definition of J_k^{rs} , $\sum_k J_k^{rs} = 0$, and $\frac{d}{d\tau} \sum_k f_k^{rs} = 0$. Thus $\sum_k f_k^{rs}(\tau) = \sum_k f_k^{rs}(0) = q_{rs}$. In summary, $f_k^{rs}(\tau > 0) \in \bar{F}_{rs}$, and \bar{F}_{rs} is an invariant set of the FIFO dynamical system. ■

An example of such invariant set is the set of a vertex equilibrium, e.g., $\{q_{rs}\mathbf{P}_k^{rs}\}$, since, if the initial state is a vertex equilibrium, the FIFO dynamical system stays there. In contrast, other dynamical systems in the literature do not have this property, and they always drive an initial state into the convex hull containing UE.

For the FIFO dynamical system (7) we have the following observation: $J_k^{rs} = 0$ when $f_k^{rs} = 0$ or $f_k^{rs} = q_{rs}$. Therefore, J_k^{rs} can be factorized into the following form:

$$J_k^{rs} = f_k^{rs} \left(q_{rs} - f_k^{rs} \right) W_k^{rs}, \quad \forall k, r, s, \quad (10)$$

which leads to a new form of the FIFO dynamical system

$$\dot{f}_k^{rs} = f_k^{rs} \left(f_k^{rs} - q_{rs} \right) W_k^{rs}, \quad \forall k, r, s. \quad (11)$$

Here we can have

$$\begin{aligned}
W_k^{rs} &= q_{rs} \left(c_k^{rs} - \frac{\sum_j c_j^{rs} f_j^{rs}}{q_{rs}} \right) / (q_{rs} - f_k^{rs}) = \left[c_k^{rs} (q_{rs} - f_k^{rs}) - \sum_{j \neq k} c_j^{rs} f_j^{rs} \right] / (q_{rs} - f_k^{rs}) \\
&= c_k^{rs} - \sum_{j \neq k} \frac{f_j^{rs}}{q_{rs} - f_k^{rs}} c_j^{rs} = c_k^{rs} - \sum_{j \neq k} \sigma_{jk}^{rs} c_j^{rs} = \sum_{j \neq k} \sigma_{jk}^{rs} (c_k^{rs} - c_j^{rs}),
\end{aligned}$$

where $\sigma_{jk}^{rs} = f_j^{rs} / (q_{rs} - f_k^{rs}), \forall j \neq k, r, s$.

We have $\sigma_{jk}^{rs} \in [0, 1]$ and $\sum_{j \neq k} \sigma_{jk}^{rs} = 1$. Obviously, when $f_k^{rs} \in [0, q_{rs})$; i.e., when $\mathbf{f}_{rs} \neq q_{rs} \mathbf{P}_k^{rs}$, W_k^{rs} is well-defined, bounded, and continuous. When $\mathbf{f}_{rs} = q_{rs} \mathbf{P}_k^{rs}$, however, W_k^{rs} may not be well-defined. An example will be given in Section 4.1.

3.2 Linearized system of the FIFO dynamical system

From equations (1) and (2), we can see that not all f_k^{rs} are independent variables. Hereafter, we only consider independent variables f_k^{rs} ($k = 1, \dots, K_{rs} - 1$), and $f_{K_{rs}}^{rs} = q_{rs} - \sum_{k=1}^{K_{rs}-1} f_k^{rs}$. The linearized form of the FIFO dynamical system around an equilibrium $\bar{\mathbf{f}}$ can be written as follows

$$\dot{\tilde{f}}_k^{rs} = \sum_{m,n} \sum_{l=1}^{K_{mn}-1} \frac{\partial(-J_k^{rs})}{\partial f_l^{mn}} \Big|_{\bar{\mathbf{f}}} \tilde{f}_l^{mn}, \quad \forall k = 1, \dots, K_{rs} - 1, \quad (12)$$

where $\mathbf{f} = \bar{\mathbf{f}} + \tilde{\mathbf{f}} \in F$. When W_k^{rs} is well-defined, we have

$$\begin{aligned}
\dot{\tilde{f}}_k^{rs} &= \sum_{m,n} \sum_l \frac{\partial}{\partial f_l^{mn}} \left(f_k^{rs} (f_k^{rs} - q_{rs}) W_k^{rs} \right) \Big|_{\bar{\mathbf{f}}} \tilde{f}_l^{mn} \\
&= (2f_k^{rs} - q_{rs}) W_k^{rs} \tilde{f}_k^{rs} + f_k^{rs} (f_k^{rs} - q_{rs}) \sum_{m,n} \sum_l \frac{\partial}{\partial f_l^{mn}} \left(W_k^{rs} \right) \Big|_{\bar{\mathbf{f}}} \tilde{f}_l^{mn}.
\end{aligned}$$

When $\bar{f}_k^{rs} = 0$, W_k^{rs} is well-defined, and we can obtain the linearized dynamical system in \tilde{f}_k^{rs} as

$$\dot{\tilde{f}}_k^{rs} = -q_{rs} W_k^{rs}(\bar{\mathbf{f}}) \tilde{f}_k^{rs}. \quad (13)$$

Thus $-q_{rs} W_k^{rs}(\bar{\mathbf{f}})$ is an eigenvalue of the linearized system (12), and the corresponding right eigenvector is \mathbf{P}_k^{rs} .

When $\bar{f}_k^{rs} = q_{rs}$, we have a vertex equilibrium $\bar{\mathbf{f}}_{rs} = q_{rs} \mathbf{P}_k^{rs}$, and W_k^{rs} may not be well-defined. We have to compute the linearized dynamical system differently. Since for $k = 1, \dots, K_{rs} - 1$

$$\begin{aligned}
J_k^{rs} &= f_k^{rs} \sum_{j=1, j \neq k}^{K_{rs}} f_j^{rs} (c_k - c_j) = f_k^{rs} \sum_{j=1, j \neq k}^{K_{rs}-1} f_j^{rs} (c_k - c_j) + f_k^{rs} f_{K_{rs}}^{rs} (c_k - c_{K_{rs}}) \\
&= f_k^{rs} \sum_{j=1, j \neq k}^{K_{rs}-1} f_j^{rs} (c_k - c_j) + f_k^{rs} (q_{rs} - \sum_{j=1}^{K_{rs}-1} f_j^{rs}) (c_k - c_{K_{rs}}),
\end{aligned}$$

we can have the linearized dynamical system of f_k^{rs} at \bar{f}_k^{rs} as

$$\dot{\tilde{f}}_k^{rs} = q_{rs} \sum_{j=1}^{K_{rs}-1} (c_j(\bar{\mathbf{f}}) - c_{K_{rs}}(\bar{\mathbf{f}})) \tilde{f}_j^{rs}. \quad (14)$$

In this case, $f_j^{rs} = 0$ for all $j \neq k$, and the linearized system for f_j^{rs} is given by equation (13). Therefore, the eigenvalue corresponding to equation (14) is $q_{rs} (c_k(\bar{\mathbf{f}}) - c_{K_{rs}}(\bar{\mathbf{f}}))$. Hereafter, we define $W_k^{rs}(\bar{\mathbf{f}})$ at a vertex equilibrium $\bar{\mathbf{f}}$ by

$$W_k^{rs}(\bar{\mathbf{f}}) = c_k(\bar{\mathbf{f}}) - c_{K_{rs}}(\bar{\mathbf{f}}). \quad (15)$$

That is, $q_{rs} W_k^{rs}(\bar{\mathbf{f}})$ is an eigenvalue of the linearized system (12).

3.3 Analysis of stability of equilibria

For a vertex equilibrium in rs -subspace, $\bar{\mathbf{f}}_{rs} = q_{rs} \mathbf{P}_k^{rs}$, from the preceding subsection, we can have the eigenvalues of the linearized system as $q_{rs} W_k^{rs}(\bar{\mathbf{f}})$ and $-q_{rs} W_j^{rs}(\bar{\mathbf{f}})$ ($j \neq k, \forall r, s$) according to equations (13) and (14). From Theorem 1, the vertex equilibrium is stable, if $W_k^{rs}(\bar{\mathbf{f}}) < 0$ and $W_j^{rs}(\bar{\mathbf{f}}) > 0$ for all $j \neq k$, and unstable if $W_k^{rs}(\bar{\mathbf{f}}) > 0$ or $W_j^{rs}(\bar{\mathbf{f}}) < 0$ for any j . That is, we have the following result for vertex equilibria.

COROLLARY 1 Stability and instability conditions of vertex equilibrium: *A vertex equilibrium in rs -subspace $\bar{\mathbf{f}}_{rs} = q_{rs} \mathbf{P}_k^{rs}$ is stable if $W_k^{rs} < 0$ for $f_k^{rs} = q_{rs}$ and $W_j^{rs} > 0$ for $j \neq k$. It is unstable if $W_k^{rs} > 0$ or $W_j^{rs} < 0$ for $j \neq k$.*

In Jin (2007), it was shown that a PUE is not stable with a perturbation method. Here we present another proof of the statement.

COROLLARY 2 Sufficient condition of UE: *For static traffic assignment problems with continuous link performance functions, a PUE is unstable for the FIFO dynamical system (7), or stable equilibria of the FIFO dynamical system are always UE.*

Proof: If $\bar{\mathbf{f}}$ is PUE of the dynamical system (7) in rs -subspace, then there exists an empty but shorter path k for some O-D pair $r-s$. Note that all used paths share the same travel time in PUE. Then, for any used path j , $c_k^{rs} < c_j^{rs}$, and $f_j^{rs} > 0$. From equation

(13), the eigenvalue of the linearized system at f_k^{rs} is $-q_{rs}W_k^{rs}(\bar{\mathbf{f}}) = -q_{rs} \sum_{j \neq k} \sigma_{jk}^{rs} (c_k^{rs} - c_j^{rs}) > 0$. From Theorem 1, the PUE is not stable. This is equivalent to saying that any stable equilibrium will be a stable UE. ■

4. EXAMPLES

In this section, we demonstrate the method of analyzing stability of UE with FIFO dynamical system and Smith's dynamical system.

4.1 A simple non-monotone traffic assignment problem

Here we study a simple non-monotone traffic assignment problem in Yang (2005, Chapter 5), where one O-D pair is connected by three paths with link performance functions as follows

$$\begin{aligned} c_1 &= 2f_1 + f_2 + 4f_3, \\ c_2 &= 4f_1 + 2f_2 + f_3, \\ c_3 &= f_1 + 4f_2 + 2f_3, \end{aligned}$$

and the O-D demand is $f_1 + f_2 + f_3 = 1$. The Jacobian matrix of this problem is

$$\begin{bmatrix} \partial c_1 / \partial f_1 & \partial c_1 / \partial f_2 & \partial c_1 / \partial f_3 \\ \partial c_2 / \partial f_1 & \partial c_2 / \partial f_2 & \partial c_2 / \partial f_3 \\ \partial c_3 / \partial f_1 & \partial c_3 / \partial f_2 & \partial c_3 / \partial f_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 2 & 1 \\ 1 & 4 & 2 \end{bmatrix},$$

the eigenvalues of which are 7 and $-\frac{1}{2}(1 \pm 3\sqrt{3}i)$. That is, the Jacobian matrix has one positive eigenvalue and two complex ones with negative real parts. Therefore, the Jacobian matrix is not positive definite, and the traffic assignment problem is non-monotone.

4.1.1 Stability analysis with the FIFO dynamical system

This non-monotone problem has the following FIFO dynamical system (f_3 can be determined by f_1 and f_2)

$$\dot{f}_1 = f_1(f_1 - 1)W_1, \quad (16)$$

$$\dot{f}_2 = f_2(f_2 - 1)W_2, \quad (17)$$

where

$$\begin{aligned} W_1 &= \frac{f_2}{1-f_1}(c_1 - c_2) + \frac{f_3}{1-f_1}(c_1 - c_3) \\ &= \frac{f_2}{1-f_1}(-5f_1 - 4f_2 + 3) + \frac{1-f_1-f_2}{1-f_1}(-f_1 - 5f_2 + 2), \end{aligned} \quad (18)$$

$$\begin{aligned} W_2 &= \frac{f_1}{1-f_2}(c_2 - c_1) + \frac{f_3}{1-f_2}(c_2 - c_3) \\ &= \frac{f_1}{1-f_2}(5f_1 + 4f_2 - 3) + \frac{1-f_1-f_2}{1-f_2}(4f_1 - f_2 - 1). \end{aligned} \quad (19)$$

Note that here W_1 is not well-defined at $(f_1, f_2) = (1, 0)$: If we denote $f_2 = \alpha(1 - f_1)$ with $\alpha \in [0, 1]$, then $\lim_{f_1 \rightarrow 1} W_1 = 1 - 3\alpha$, which is not unique. Similarly, W_2 is not well-defined at $(f_1, f_2) = (0, 1)$. Therefore, we use equation (15) to define $W_1(1, 0) = c_1(1, 0) - c_3(1, 0)$ and $W_2(0, 1) = c_2(0, 1) - c_3(0, 1)$.

As shown in Figure 1, this system has totally four equilibria including one UE and three PUE: $(f_1, f_2) = \text{UE1} (\frac{1}{3}, \frac{1}{3})$, E2 (0, 0), E3 (0, 1), and E4 (1, 0). Note that all PUE are vertex equilibria. When $W_1 = 0$, we find a separation curve

$$f_{2*} = \frac{1}{2}(4 - f_1 - \sqrt{8 + 4f_1 - 3f_1^2}), \quad 0 \leq f_1 \leq 1. \quad (20)$$

When $W_2 = 0$, we find another separation curve

$$f_{1*} = \frac{1}{2}(-2 - f_2 + \sqrt{-3f_2^2 + 4f_2 + 8}), \quad 0 \leq f_2 \leq 1. \quad (21)$$

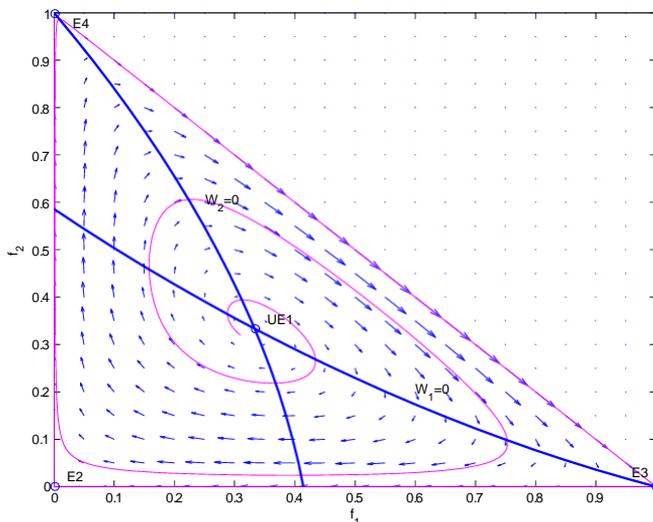


FIGURE 1: Stability analysis with the FIFO dynamical system for a non-monotone traffic assignment problem from (Yang, 2005)

In the region above the first separation curve, $W_1 < 0$, and f_1 increases; in the region below, $W_1 > 0$, and f_1 decreases. In the region left to the second separation curve, $W_2 < 0$, and f_2 increases; in the region right, $W_2 > 0$, and f_2 decreases. From the figure we can also see that W_1 can be positive or negative at (1, 0) and, therefore, is not well-defined. We can have similar observation on W_2 and (0, 1).

From equations (13) and (14), we have the two eigenvalues of the linearized systems around E2, E3, and E4 as $(-W_1(0, 0), -W_2(0, 0)) = (-2, 1)$, $(-W_1(0, 1), W_2(0, 1)) = (1, -2)$, $(W_1(1, 0), -W_2(1, 0)) = (1, -2)$ respectively. Therefore, as shown in Figure 1, all three vertex equilibria are unstable saddle points. To determine the stability of UE1, we can

use linearization method at this point for equations (16) and (17) as follows ($\tilde{f}_1 = f_1 - \frac{1}{3}$ and $\tilde{f}_2 = f_2 - \frac{1}{3}$)

$$\dot{\tilde{f}}_1 = \frac{2}{3}\tilde{f}_1 + \tilde{f}_2, \quad (22)$$

$$\dot{\tilde{f}}_2 = -\tilde{f}_1 - \frac{1}{3}\tilde{f}_2, \quad (23)$$

the eigenvalues of which are $\frac{1}{6}(1 \pm 3\sqrt{3}i)$. Therefore the UE is an unstable spiral for the linearized system, and the boundary triangle shown in Figure 1 is a limit cycle (Strogatz, 1994, Chapter 7). Note that the stability of the UE is consistent with that from Yang (2005), but the limit cycles are not the same.

4.1.2 Stability analysis with Smith's dynamical system

Smith's dynamical system for the non-monotone traffic assignment problem (16) and (17) can be written as follows:

$$\begin{aligned} \dot{f}_1 &= -f_1(c_1 - c_2)_+ - f_1(c_1 - c_3)_+ + f_2(c_2 - c_1)_+ + f_3(c_3 - c_1)_+ \\ &= \begin{cases} -f_1^2 - f_2^2 - f_1f_2 + 3f_1 + 4f_2 - 2, & \text{in regions 1 and 2} \\ f_1^2 + 4f_2^2 + 10f_1f_2 - 2f_1 - 3f_2, & \text{in region 3} \\ 6f_1^2 + 9f_1f_2 - 5f_1, & \text{in regions 4 and 5} \\ 4f_1^2 - 5f_2^2 - 2f_1f_2 + 7f_2 - 2, & \text{in region 6} \end{cases} \end{aligned} \quad (24)$$

$$\begin{aligned} \dot{f}_2 &= -f_2(c_2 - c_1)_+ - f_2(c_2 - c_3)_+ + f_1(c_1 - c_2)_+ + f_3(c_3 - c_2)_+ \\ &= \begin{cases} 4f_1^2 - 5f_2^2 - 2f_1f_2 - 5f_1 + 3f_2 + 1, & \text{in region 1} \\ -3f_2^2 - 9f_1f_2 + 4f_2, & \text{in regions 2 and 3} \\ -5f_1^2 + f_2^2 - 8f_1f_2 + 3f_1 + f_2, & \text{in region 4} \\ -f_1^2 - f_2^2 - f_1f_2 - 2f_1 + 1, & \text{in regions 5 and 6} \end{cases} \end{aligned} \quad (25)$$

where the six regions in the domain of (f_1, f_2) are shown in Figure 2.

From Smith (1984b) we know that the dynamical system has only one equilibrium, $UE1 = (\frac{1}{3}, \frac{1}{3})$. However, it is relatively complicated to compute the equilibrium from the dynamical system above and hard to judge its stability analytically. To determine the stability of UE1, we can use linearization method at this point for equations (24) and (25)² as follows ($\tilde{f}_1 = f_1 - \frac{1}{3}$ and $\tilde{f}_2 = f_2 - \frac{1}{3}$)

$$\dot{\tilde{f}}_1 = 2\tilde{f}_1 + 3\tilde{f}_2, \quad (26)$$

$$\dot{\tilde{f}}_2 = -3\tilde{f}_1 - \tilde{f}_2, \quad (27)$$

the eigenvalues of which are $\frac{1}{6}(1 \pm 3\sqrt{3}i)$. Therefore UE1 is an unstable spiral for the linearized system, as shown in Figure 2. The stability of UE determined by Smith's dynamical system is consistent with that by the FIFO dynamical system.

² It is not hard to verify that Smith's dynamical system (24) and (25) is differentiable at UE1.

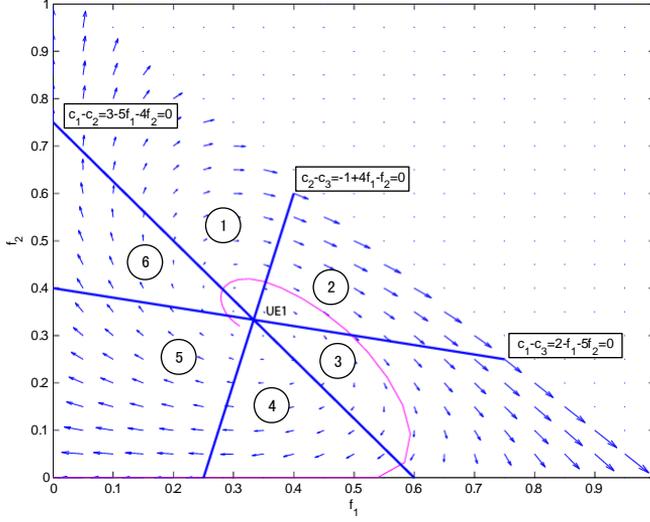


FIGURE 2: Stability analysis with Smith's dynamical system for a non-monotone traffic assignment problem from Yang (2005)

4.2 A non-monotone multi-class traffic assignment problem

As shown in Dafermos (1972), in multi-class traffic assignment problems, classes can be treated similarly as O-D pairs. That is, if there are N_{rs} O-D pairs and N_d classes, then the problem is equivalent to a network with $N_{rs}N_d$ O-D pairs. Thus, the FIFO dynamical system for link k of O-D pair r - s and class d can be directly extended for multi-class problems as

$$\dot{f}_k^{rs,d} = f_k^{rs,d} \left(f_k^{rs,d} - q_{rs,d} \right) W_k^{rs,d}, \quad \forall k, r, s, d. \quad (28)$$

Here we consider the example given in Netter (1972), where two classes of users share two routes connecting one O-D pair as shown in Figure 3. The network conditions are given in the following (Here rs is omitted since there is only one O-D pair, and the subscript and superscript stand for path and class respectively)

$$\begin{aligned} c_1^1 &= 0.5f_1^1 + 5f_1^2 + 6, & c_2^1 &= 0.5f_2^1 + 3f_2^2 + 10, \\ c_1^2 &= 0.3f_1^1 + 0.6f_1^2 + 0.8, & c_2^2 &= 0.2f_2^1 + 0.4f_2^2 + 2, \\ f_2^1 &= 16 - f_1^1, & f_2^2 &= 4 - f_2^2. \end{aligned}$$

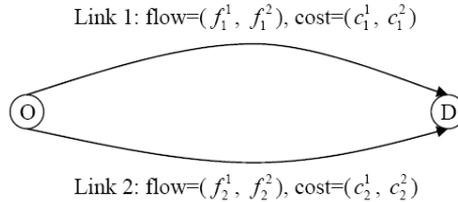


FIGURE 3: A two-class, two-route road network

The Jacobian matrix of this problem has four eigenvalues, $\frac{1}{20}(11 \pm \sqrt{601})$ and $\frac{1}{20}(9 \pm \sqrt{241})$, and is not positive definite. Thus the corresponding traffic assignment problem is non-monotone.

4.2.1 Stability analysis with the FIFO dynamical system

This non-monotone problem has the following FIFO dynamical system (f_1^1 and f_1^2 can be determined by f_1^1 and f_2^2 respectively)

$$\dot{f}_1^1 = -J_1^1 = f_1^1(f_1^1 - 16)W_1^1, \quad \dot{f}_2^2 = -J_2^2 = f_2^2(f_2^2 - 4)W_2^2, \quad (29)$$

where

$$W_1^1 = c_1^1 - c_2^1 = f_1^1 - 8f_2^2 + 8, \quad W_2^2 = c_2^2 - c_1^2 = -0.5f_1^1 + f_2^2 + 2. \quad (30)$$

Note that here W_1 and W_2 are both well-defined in the considered domain. This system has totally five equilibria including three UE and two PUE: $(f_1^1, f_2^2) = \text{UE1}(0, 0)$, $\text{UE2}(16, 4)$, $\text{UE3}(8, 2)$, $\text{E4}(0, 4)$, and $\text{E5}(16, 0)$. Except $\text{UE3}(8, 2)$, the other four equilibria are all vertex equilibria.

From equations (13) and (14), two eigenvalues of the linearized systems around UE1 , UE2 , E4 , and E5 are $(-16W_1^1(0,0), -4W_2^2(0,0))$, $(16W_1^1(16,4), 4W_2^2(16,4))$, $(-16W_1^1(0,4), 4W_2^2(0,4))$, and $(16W_1^1(16,0), -4W_2^2(16,0))$ respectively. The signs of (W_1^1, W_2^2) in four regions are shown in Figure 4b, from which we can see that UE1 and UE2 are stable, but E4 and E5 are not. Thus UE1 and UE2 are stable sinks, and E4 and E5 unstable sources of the FIFO dynamical system (29). To determine the stability of the third UE, we can linearize (29) at $(8, 2)$ and obtain $(\tilde{f}_1^1 = f_1^1 - 8$ and $\tilde{f}_2^2 = f_2^2 - 2)$

$$\dot{\tilde{f}}_1^1 = -64(\tilde{f}_1^1 - 8\tilde{f}_2^2), \quad \dot{\tilde{f}}_2^2 = 2(\tilde{f}_1^1 - 2\tilde{f}_2^2), \quad (31)$$

the eigenvalues of which are $2(-17 \pm \sqrt{481})$. Since the eigenvalues are of different signs, $\text{UE3}(8, 2)$ is an unstable saddle point for the dynamical system (29) (Strogatz, 1994, Chapter 5). Shown in Figure 4a are the vector field and solution trajectories starting from four initial values: $(4, 2)$, $(4, 3)$, $(12, 1)$, and $(12, 2)$. Comparing this figure with Figure 1 of Netter (1972), we observe the same stability results: Trajectories with initial states in region I are attracted to UE1 , those with initial states in region II are attracted to UE2 , and those with initial states in regions III and IV are first attracted to UE3 and then penetrate into either region I or region II. Moreover, from numerical solutions of the dynamical system (29), we can obtain a separation curve, shown by the dashed curve in Figure 4b. Trajectories starting with initial values above the separation curve converge to UE2 , and those with initial states below converge to UE1 . The existence of such a separation curve in Figure 4b is consistent in principle with the numerical results in (Wynter, 1995; Marcotte and Wynter, 2004), where the solution process also has similar stability properties. Note that, in Netter (1972), the stability results are from the price adjustment process in Appendix 2 of Netter (1972).

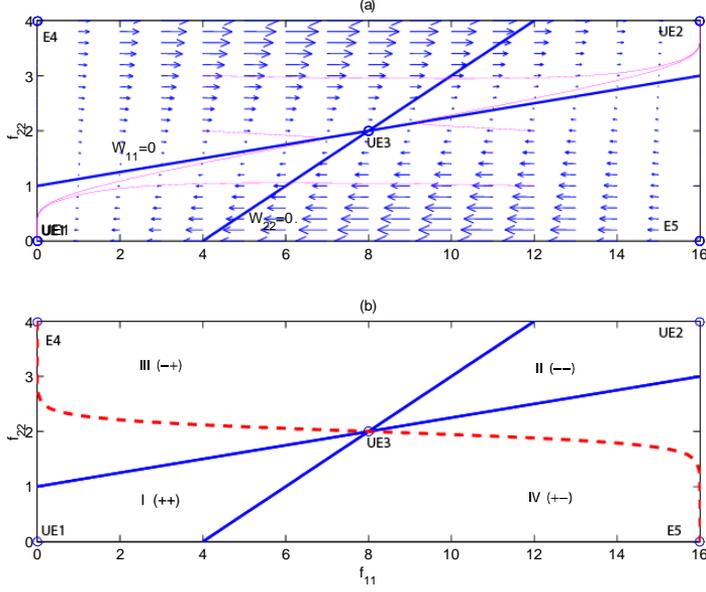


FIGURE 4: Stability property of UE for a multi-class traffic assignment problem from Netter (1972)

4.2.2 Stability analysis with Smith's dynamical system

Smith's dynamical system for the non-monotone multi-class problem can be written as follows:

$$\dot{f}_1^1 = \begin{cases} -f_1^1 W_1^1, & W_1^1 > 0 \\ -(16 - f_1^1) W_1^1, & W_1^1 < 0 \end{cases}, \quad \dot{f}_2^2 = \begin{cases} -f_2^2 W_2^2, & W_2^2 > 0 \\ -(4 - f_2^2) W_2^2, & W_2^2 < 0 \end{cases} \quad (32)$$

where W_1^1 and W_2^2 are defined in equation (30). For this dynamical system, there are only three equilibria: UE1, UE2, and UE3. We can see that the first two equilibria are stable. At UE3, although the right hand-side of Smith's dynamical system is a piece-wise function, it is differentiable, and we can obtain the following linearized system ($\tilde{f}_1^1 = f_1^1 - 8$ and $\tilde{f}_2^2 = f_2^2 - 2$)

$$\dot{\tilde{f}}_1^1 = -8(\tilde{f}_1^1 - 8\tilde{f}_2^2), \quad \dot{\tilde{f}}_2^2 = \tilde{f}_1^1 - 2\tilde{f}_2^2,$$

which has two eigenvalues of opposite signs. Therefore, the stability of UE determined by Smith's dynamical system is consistent with that by the FIFO dynamical system.

5. CONCLUSION

In this paper, based on a so-called FIFO dynamical system model of the traffic assignment problem, we proposed a new approach to analyzing stability of UE for static traffic assignment problems and applied the approach to discussing the stability properties of user equilibria of two examples of non-monotone traffic assignment problems, whose UE can be an unstable spiral, a stable sink, or an unstable saddle point.

For both problems, the stability of UE determined by the FIFO dynamical system is consistent with that by Smith's dynamical system (Smith, 1984b). We also demonstrate that the former is simpler than the latter, which has piece-wise right-hand sides. We expect that the simplicity of the new approach is more apparent for traffic assignment problems of more complicated networks, including static traffic assignment problems with variable demands. Especially, the new approach could be extended for analyzing stability of UE for dynamic traffic assignment problems, which would probably be non-monotone due to complicated traffic dynamics and capacity constraints of road networks. In this sense, this study could better our understanding of stability of dynamic user equilibria and help to improve the analysis and design of transportation networks.

In the literature, relaxation or diagonal methods have been used to compute UE of asymmetric traffic assignment problems (Dafermos, 1982b; Fisk and Nguyen, 1982; Smith, 1984a; Nguyen and Dupuis, 1984; Marcotte and Guélat, 1988; Patriksson, 1993; Marcotte and Wynter, 2004; Nagurney and Zhang, 1996, 1997). Due to their internal connection with the mathematical programming formulation, these methods cannot guarantee convergence to unstable UE. In the follow-up studies, we are interested in computing unstable UE with the FIFO dynamical system for such problems. Another interesting problem is to discuss the stability of UE for monotone traffic assignment problems with the new approach. In the future, we will also be interested in stability of user equilibria in dynamic traffic assignment problems (Huang and Lam, 2002; Szeto and Lo, 2006).

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