Uncertainty and the timing of an urban congestion relief investment

The no-land case[1]

Jean-Daniel M. Saphores, Corresponding Author
Department of Planning, Policy, and Design, Department of Economics,
and Department of Civil & Environmental Engineering,
University of California, Irvine, CA 92697.
Phone: (949) 824 7334; E-mail: saphores@uci.edu,

and

Marlon G. Boarnet
Department of Planning, Policy, and Design, and Department of Economics
University of California, Irvine, CA 92697.
Phone: (949) 824 7695; E-mail: mgboarne@uci.edu.

Running title: Reducing congestion under uncertainty

[1] We gratefully acknowledge helpful comments from Ken Small and Ami Glazer. We are also very thankful to the editor (Jan Brueckner) and two anonymous referees whose excellent suggestions helped improve this paper significantly. We are responsible for all remaining errors.

Abstract

We analyze the impact of population uncertainty on the socially optimum timing of a congestion-relief project in a linear monocentric city with fixed boundaries, where congestion pricing cannot be implemented. This project requires time to bear fruit but no urban land. Under certainty, we show that utility maximization is roughly equivalent to a standard benefit-cost analysis (BCA). Under uncertainty, we derive an explicit optimal threshold for relieving congestion when the urban population follows a geometric Brownian motion. If the time to implement the project is short, we show analytically that deciding on the timing of congestion relief based on a BCA could lead to acting prematurely; the reverse holds if project implementation is long and uncertainty is large enough.

Keywords: transportation infrastructure; congestion; Taylor expansion; uncertainty; irreversibility; investment lag; real options.

JEL codes: D61, D81, R42.

I. INTRODUCTION

When to invest in transportation infrastructure is an essential question for transportation agencies, elected officials, and private transportation entrepreneurs alike. Leaving demand unmet for too long can entail large social costs as congestion increases and pollution builds up. On the other hand, investing too early may create large social and/or private costs if underused capacity creates revenue shortfalls. Recent examples of private highway projects that initially faced smaller than anticipated demand include the Dulles Toll Road extension in suburban Virginia and the San Joaquin Hills Transportation Corridor in Orange County, California. Similar timing problems can also plague public projects, although underused public facilities are not as commonly publicized. More generally, timing is an issue for any type of congestible infrastructure.

Surprisingly, however, the urban economics literature is almost silent on the timing under uncertainty of congestion-relief investments. A number of authors analyze the allocation of land for roadways, but they usually adopt a deterministic framework with a fixed population. Building on the seminal papers of Mills and de Ferranti [27], who focus on congestion facing commuters, and Solow and Vickrey [39], who are concerned with congestion and goods shipment, there have been many important contributions to this literature, including Arnott [1], Dixit [10], Hartwick [15], Hochman [16], Kanemoto [21] [22], Legey, Ripper, and Varaiya [23], Livesey [24], Mumy [30], Oron, Pines, and Sheshinski [32], Pines and Sadka [34], Robson [35], Sheshinski [38], and Solow [40] [41].

A few related papers allow for deterministic urban growth; recently, Braid [5] considered a government planner with perfect foresight concerned with allocating land between housing and roadways at the urban boundary. Another exception is Szymanski [43], who examined differences in the timing of infrastructure investment between a welfare maximizing public agency and a profit maximizing private firm.

This paper extends this literature by analyzing the impact of population uncertainty on the socially optimum timing of an urban congestion-relief investment that requires no urban land but takes some time to implement. We consider a linear monocentric city where congestion pricing is too costly to implement, so residents incur both transportation and congestion costs. Following Solow [40], the area of our city is fixed by natural or political boundaries. Our city is also implicitly part of a system of open cities (e.g., see Frame [14]), so when congestion is too high, people move out, which is akin to bounding the city population from above. It is also bounded from below because of unique factors linked to location, politics, or history. We formulate the corresponding optimal stopping problem using real options (see Dixit and Pindyck [12] and the references herein) and rely on Taylor expansions (Sprecher [42]) to simplify differences in indirect utility.

When the city population increases at a constant rate, we derive a utility-based rule of thumb for evaluating a congestion relief project and show that a standard benefit-cost analysis (BCA) is approximately correct. When the city population follows a geometric Brownian motion (GBM), we obtain an explicit decision rule provided the impact of the population barriers can be ignored. We show that basing the decision to relive congestion on a BCA could lead to acting prematurely when the time needed to implement the project is short, and to acting too late otherwise if uncertainty is large enough. This result reflects the tradeoff between waiting and investing to reduce congestion in the presence of a time lag: waiting for a larger population increases the value of congestion relief and postpones the flow of project costs, but it also increases the present value of the disutility from congestion until the project is complete, so the time needed to implement a project is important. If this time lag is long enough and if uncertainty is high enough, it becomes optimal to invest in congestion relief earlier rather than later. In sum the effect of uncertainty is ambiguous because it affects both the benefits and the costs of waiting. Using numerical methods, Bar-Ilan and Strange [3] obtain similar results in their study of the impact of time lags on land development under uncertainty.

In general, finding the optimum timing of congestion-relief investments, especially in infrastructure, is inherently difficult for at least three reasons. First, the net social benefits from such investments are stochastic because of economic cycles, competition from other projects, or technological change. The most commonly acknowledged uncertainties are in project costs and user demand. We focus here on population uncertainty because it can often be used as a proxy for other forms of uncertainty, and also because it is often considerable over the time horizon required for major projects. Indeed, Johnson [19] remarks that reputable population projections for California in 2025 vary from a low of 41.5 million to a high of 52 million because of different assumptions about net U.S. to California migration in the future.

Second, the decision to invest in infrastructure is largely irreversible: if it turns out that a new transportation project is not needed, only a fraction of the initial investment typically can be recovered, and environmental impacts may be felt for years. The role of irreversibility is amplified by the large amounts of capital usually required by infrastructure projects.

Third, there is often a considerable lag between the start of a transportation project and its completion. Coupled with uncertainty, this lag can complicate the decision-making process, as illustrated by Bar-Ilan and Strange for general investment decisions [2] and land conversion [3].

The transportation economics literature has long been concerned with benefit-cost assessments of transportation projects (e.g., see Mohring [28] or Mohring and Harwitz [29]), but it does not seem to have incorporated insights from the theory of investment under uncertainty (see Dixit and Pindyck [12] and the references therein), which shows that a standard benefit-cost analysis of an irreversible and uncertain investment can be seriously biased as it ignores the value of the flexibility to make a binding decision. The real options approach has been fruitfully applied to land conversion (e.g., see [3], [6], [7], or [8]), but it has not received much attention for public investments.

This paper is organized as follows. In the next section, we develop a framework for finding the socially optimal timing of a congestion relief investment whose implementation takes time but no urban land. In Section III, we find a utility-based rule of thumb for assessing the worth of such an investment when the urban population increases at a constant rate. In Section IV, we derive an explicit decision rule for investing in congestion relief when the city population follows a GBM. We present brief conclusions in Section V. Table 1 summarizes key notation.

II. THE MODELING FRAMEWORK

In this section, we adapt standard features of the monocentric model to a real options framework. After discussing some key assumptions, we derive the flow of indirect utility of city residents before, during, and after a project designed to relieve congestion. Expressions related to these phases are respectively indexed by the subscripts “0,” “01,” and “1.” These steps are necessary to formulate the objective of a welfare-maximizing social planner.

A. City Geography and Population

We consider an open, rectangular city located on a homogeneous plain. Employment and production are located in the central business district (CBD), which is a linear strip of land of width B that separates the city in two equal rectangular halves. As suggested by Braid [5], the CBD could be located along a river that bisects the city. We normalize the aerial distance of the CBD to the city edge to 1 and denote by $h \in [0,1]$ the aerial distance between a point in the city and the CBD (see Figure 1). City residents are identical; as they commute, they incur both transportation and congestion costs.

The total city population, denoted by X_t, changes stochastically over time according to the regular autonomous diffusion process:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dw_t. \tag{1}$$

In Equation (1), $\mu(x)$ and $\sigma(x)>0$ are continuous; dt is an infinitesimal time increment; and dw_t is an increment of a standard Wiener process (Dixit and Pindyck [12]). Throughout this paper, x is a realization of the random variable X_t.

In general, a changing urban population has two consequences for land use: it affects density and the extent of the urban area. To abstract from the complexities of land conversion (e.g., see Bar-Ilan and Strange [3], Cappoza and Helsley [6], Cappoza and Sick [7], or Clarke and Reed [8]), we assume that the city limits are set either by natural or political boundaries, as in Solow [40], or by growth boundaries (as in Portland, Oregon). In any case, the area of a monocentric city is necessarily bounded because its residents cannot live beyond the point where all of their income must be spent on transportation.

A bounded urban area then implies an upper limit on the urban population in a closed system of open cities, as in Frame [14] or Henderson and Ioannides [17], or in a network of open cities where new urban areas are created when diseconomies of scale grow too large. We therefore assume that the city population is bounded by an upper reflecting barrier $\overline{x}$, so X_t is reflected downwards when it reaches $\overline{x}$. Inter-city competition also may create a lower reflecting barrier $\underline{x}$ (it reflects X_t upwards) if the drop in negative urban externalities (crime, congestion, pollution) from a population drop outweighs the corresponding loss of agglomeration benefits. Alternatively, the city population may be bounded from below because of tourism (linked to unique historical sites or a pleasant climate), a large administrative sector (in a capital city), strategic infrastructure assets (a deepwater seaport), or a combination of these factors.

Population barriers simplify our analysis: with a lower barrier, we can ignore the decision to abandon infrastructure as the flow of project costs stays small compared to the flow of individual income. Likewise, an upper barrier guarantees that the flows of transportation and congestion costs remain a small fraction of the income flow. We can then derive simplified expressions of the indirect utility of urban residents using simple Taylor expansions.

B. Demand for Land, Congestion, and Indirect Utility

For simplicity, the flow of individual income is assumed constant and we normalize it to one. City residents derive utility from land and from a numeraire, but they also need to pay for transportation and congestion costs. We do not adopt here the traditional bottleneck congestion formulation pioneered by Vickrey [47], which defines congestion on a single road segment; getting explicit land rents with that approach would require imposing constant residential densities (as in Braid [5]) and defining an explicit road network. Instead, we assume that the *average flow of congestion costs incurred over the urban transportation network per unit of time* is proportional to a function of the total city population and to the distance to the CBD.

The per-unit-of-time budget constraint of a city resident living at a distance $h \in [0,1]$ from the CBD when the total city population is x can then be written

$$g + R_0(x,h)L + (\omega_0(x) + \lambda)h = 1, \tag{2}$$

where:

- g is the fraction of the time flow of individual income spent on the composite good;
- $R_0(x,h)$ is the fraction of the flow of individual income spent on rent per unit area;
- L is the area of land rented; moreover,
- $\omega_0(x)$ and λ are the fractions of the flow of individual income spent respectively on congestion and transportation by a resident living at the city edge (h=1).

We suppose that the flow of utility of an urban resident can be described by the logarithm of a Cobb-Douglas utility function

$$U(g,L) = a\ln(g) + (1-a)\ln(L), \tag{3}$$

where $a \in (0,1)$ is the utility elasticity of the numeraire. Following Varian [46], the demand functions for land and for the numeraire of a city resident living at a distance h from the CBD are then respectively given by

$$\begin{cases} L(R_0(x,h),1) = (1-a)R_0(x,h)^{-1}\left[1-(\omega_0(x)+\lambda)h\right], \\ g(R_0(x,h),1) = a\left[1-(\omega_0(x)+\lambda)h\right]. \end{cases} \tag{4}$$

At equilibrium, all city residents have the same utility. Equating utility flows at the CBD and at a distance h from the CBD tells us how land rent changes as we move away from the city center:

$$R_0(x,h) = R_0(x,0)\left[1-(\omega_0(x)+\lambda)h\right]^{\nu+1}, \tag{5}$$

where ν is the ratio of the utility elasticity of the numeraire to the utility elasticity of land:

$$\nu = \frac{a}{1-a}. \tag{6}$$

Now let $N_0(x,h)$ designate the number of urban residents who live within a distance h of the CBD when total city population is x. In the two parallel strips of length B and thickness dh at distance h from the CBD, there are $N_0(x,h+dh)-N_0(x,h)=\frac{\partial N_0(x,h)}{\partial h}dh$ city residents (Figure 1); multiply this number by $L(R_0(x,h),1)$ from Equation (4) to get the total demand for land there. Since the corresponding land supply is $2Bdh$, equilibrium in the land market requires:

$$\frac{\partial N_0(x,h)}{\partial h} = \frac{2(1+\nu)\ B\ R_0(x,h)}{1-\left(\omega_0\left(x\right)+\lambda\right)h}. \tag{7}$$

When we introduce Equation (5) into Equation (7), integrate $\frac{\partial N_0(x,h)}{\partial h}$ over h between 0 and 1, and equate it to x, we find that the flow of unit area land rents at the CBD is

$$R_0(x,0) = \frac{x}{2B}\frac{\omega_0(x)+\lambda}{1-\left[1-\omega_0(x)-\lambda\right]^{\nu+1}}. \tag{8}$$

In equilibrium, all residents have the same flow of indirect utility $V_0(x)$, so it equals the flow of indirect utility at the CBD:

$$V_0(x) = a\ln\left(a\right)+(1-a)\ln\left(\frac{(1-a)2B}{x}\frac{1-\left[1-\omega_0(x)-\lambda\right]^{\nu+1}}{\omega_0(x)+\lambda}\right). \tag{9}$$

C. Investing in Congestion Relief

We suppose that congestion pricing is too complex or politically impossible to implement, so we adopt a second-best framework where a social planner decides on the timing of a congestion relief project that lowers $\omega_0(x)$ to $\omega_1(x)$. Implementing this project requires Δ units of time but essentially no urban land, as for elevated (road or metro) or underground (metro or road tunnel) transportation systems.

Total project costs C consist of construction and all future maintenance and operation costs (M&O), which we lump together. If C is annualized over an infinite time horizon and shared equally among city residents, the ratio of the flow of interests on project costs to the flow of total city income equals $\Omega(x) \equiv \dfrac{\rho C}{x}$ (recall that individual income is normalized to be 1), where ρ is the social planner's discount rate.

After project completion, the budget constraint of an urban resident changes from (2) for two reasons: first, congestion is reduced, and second, a fraction $\Omega(x)$ of the flow of individual income goes toward paying for the project. The budget constraint (2) then becomes

$$g + R_1(x,h)L + (\omega_1(x) + \lambda)h = 1 - \Omega(x). \tag{10}$$

To derive the new demand functions from (4), replace $\omega_0(x)$ with $\omega_1(x)$, $1-\left(\omega_0\left(x\right)+\lambda\right)h$ with $1-\Omega(x)-\left(\omega_1\left(x\right)+\lambda\right)h$, and $R_0(x,h)$ with $R_1(x,h)$. Using the same logic as above, we find

$$R_1(x,h) = \frac{x}{2B}\frac{\left[\omega_1(x)+\lambda\right]\left[1-\Omega(x)-\left(\omega_1\left(x\right)+\lambda\right)h\right]^{\nu+1}}{\left[1-\Omega(x)\right]^{\nu+1}-\left[1-\Omega(x)-\omega_1(x)-\lambda\right]^{\nu+1}}, \tag{11}$$

so, after simplifications, the post-construction flow of indirect utility of a city resident is

$$V_1(x) = a\ln\left(a\right) + (1-a)\ln\left(\frac{(1-a)2B}{x}\frac{\left[1-\Omega(x)\right]^{\nu+1} - \left[1-\Omega(x)-\omega_1(x)-\lambda\right]^{\nu+1}}{\omega_1(x)+\lambda}\right). \quad (12)$$

During construction (subscript "01"), city residents must pay for the project, but there is no congestion relief. The budget of a city resident is thus given by Equation (10) after changing $\omega_1(x)$ to $\omega_0(x)$. Likewise, the land rent per unit area, $R_{01}(x,h)$, and the flow of indirect utility, $V_{01}(x)$, can be obtained from Equations (11) and (12) by replacing $\omega_1(x)$ with $\omega_0(x)$.

D. The Social Planner's Objective Function

We can now express the objective of a welfare-maximizing social planner, who is looking for when, if ever, to invest in an urban congestion-relief project. This is a standard stopping problem (see Dixit and Pindyck [12]). Indeed, when the city population is small, congestion is low and per-capita costs of transportation infrastructure improvements are high, so it is best to wait; this defines the "waiting region." Conversely, when the city population is large, congestion is high and the per-capita costs of infrastructure improvements are low, so it is optimal to invest immediately; this defines the "stopping region." Therefore, the social planner is looking for x^*, the population threshold that separates the waiting region from the stopping region.

Let us assume that the present value of the aggregate expected utility of city residents with the project exceeds the present value of their aggregate expected utility without the project in order to derive the first-order necessary condition satisfied by $x^* \in (\underline{x}, \overline{x})$. Using a financial analogy at the heart of the real options approach, the social planner holds an option, which gives her the possibility (not the obligation) to invest in congestion relief. This option depends only on the total city population (and on parameters defining the investment problem), so we denote it by $\varphi(x)$. A standard result in real options theory (see Chapter 5 in Dixit and Pindyck [12]) is that over $[\underline{x}, x^*]$, $\varphi(x)$ satisfies the equilibrium condition

$$\rho\varphi(x) = \mu(x)\frac{\partial\varphi(x)}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2\varphi(x)}{\partial x^2}. \quad (13)$$

The left side of Equation (13) is the normal return needed to hold the option to relieve congestion. Its right side is derived by applying Ito's lemma ([12] p. 79) to the expectation of an increment of $\varphi(x)$; it represents the actual appreciation of the option (capital gains). Moreover, since X_t is reflected at $\underline{x}$, we know from Dixit [11] that $\varphi(x)$ satisfies the boundary condition

$$\frac{\partial\varphi(x)}{\partial x}\Big|_{x^*} = 0. \quad (14)$$

Equation (13) is a second-degree ordinary differential equation; with Equation (14), it defines $\varphi(x)$ within a constant that needs to be found jointly with x^*. So, two more conditions are needed.

To find the first condition, we note that by definition of x^*, the social planner is indifferent between two assets at x^*. The first asset is the sum of the present value of the expected utility of city residents without the project, plus the value of the option to invest in congestion-relief. The second asset is simply the present value of the expected utility of city residents with the project. This leads to the "continuity condition":

$$\varphi(x^*) + E_{x^*}\int_0^{+\infty} X_t V_0(X_t)e^{-\rho t}dt = E_{x^*}\left\{\int_0^{\Delta} X_t V_{01}(X_t)e^{-\rho t}dt + \int_{\Delta}^{+\infty} X_t V_1\left(X_t\right)e^{-\rho t}dt\right\}. \quad (15)$$

Now bring all integrals to the right side of (15), add and subtract $E_{x^*} \int_{\Delta}^{+\infty} X_t V_{01}\left(X_t\right) e^{-\rho t} dt$ from the right side, and simplify, to get

$$\varphi(x^*) = P(x^*). \tag{16}$$

where $P(x)$ is the present value of the expected utility changes from the project when it starts:

$$P(x) = E_x \left\{ \int_0^{+\infty} X_t [V_{01}(X_t) - V_0(X_t)] e^{-\rho t} dt + \int_{\Delta}^{+\infty} X_t [V_1(X_t) - V_{01}(X_t)] e^{-\rho t} dt \right\}. \tag{17}$$

Equation (17) partitions project costs (first integral) and benefits (other terms). Indeed, $V_{01}(X_t) - V_0(X_t)$ is the flow of utility loss of a city resident during the implementation of the project: he is paying for the project without enjoying any congestion relief. The first integral on the right side of $P(x)$ goes from 0 to $+\infty$ although the implementation of the project lasts for a period Δ, so the second integral on the right of $P(x)$ is a correction for project benefits. Indeed, $V_1(X_t) - V_{01}(X_t)$ is the flow of utility gain for a city resident that follows project completion: project costs are still paid but congestion is now reduced.

The second condition, called “smooth-pasting” in the real options literature (Dixit [11]), requires a smooth transition between the values of the two assets to prevent arbitrage, so that:

$$\frac{d\varphi(x)}{dx}\Big|_{x=x^*} = \frac{dP(x)}{dx}\Big|_{x=x^*} . \tag{18}$$

To find the first-order necessary condition for an interior solution for x^*, we proceed as in Dixit, Pindyck and Sødal [13]: after dividing Equation (18) by Equation (16), we multiply both sides by x^* to get:

$$\varepsilon^{\varphi} \equiv \left(\frac{x}{\varphi(x)} \frac{d\varphi(x)}{dx} \right)\Big|_{x=x^*} = \varepsilon^{P} \equiv \left(\frac{x}{P(x)} \frac{dP(x)}{dx} \right)\Big|_{x=x^*} . \qquad (19)$$

ε^{φ} is the population elasticity at x^* of the option to relieve congestion, and ε^{P} is the population elasticity at x^* of $P(x)$ (see Equation (17)), the present value of expected utility changes from the project. Equation (19) expresses the tradeoff between waiting and investing to reduce congestion: waiting for a larger population increases the value of congestion relief and postpones the flow of project costs, but it also increases the present value of the disutility from congestion until the project is complete. Equation (19) can be solved numerically if the analytical expression of the probability density function of X_t is known (see Saphores [37]). In the rest of this paper, however, we focus on deriving analytical solutions for special cases.

E. Taylor Approximations of Changes in Indirect Utility

To obtain further results, it is useful at this point to get a sense of how the flows of congestion costs and infrastructure project costs compare to the flow of income. According to the Texas Transportation Institute [45], annual congestion delay costs per driver averaged \$1,590 in the largest metropolitan statistical areas studied. The 2000 median household income was \$42,228, so a plausible value of the average flow of congestion as a fraction of the flow of income ($\omega_0(x)$ in our notation) is 0.038 (\$1,590 / \$42,228). Pure transportation costs are of the same order. In addition, a review of recent highway or rail transit-corridor investments in U.S. metropolitan areas ([4], [18], [25], [33], [36] and [49]) shows that, excluding maintenance, $\Omega(x)$ (see Table 1 for a definition) ranges from 3.2×10^{-5} to 5.1×10^{-4} (for ρ=4%). Even if elevated or underground transportation systems as well as annualized maintenance costs increase these numbers by two orders of magnitude, $\Omega(x)$ is still much smaller than 1. Details are available from the authors.

Now, if the lower population barrier $\underline{x}$ is sufficiently high, the ratio of the flow of project costs to the flow of total income is small compared to 1 (i.e., «1) for all values of $X_t \in [\underline{x}, \overline{x}]$. Likewise, if the upper barrier $\overline{x}$ is low enough, the flows of transportation and congestion costs remain a small fraction of the income flow over $[\underline{x}, \overline{x}]$. In the rest of this paper, we suppose these assumptions hold, so simple first-order Taylor expansions of differences in the indirect utility of urban residents are valid over $[\underline{x}, \overline{x}]$, irrespective of the value of the other model parameter. Taylor expansions are powerful tools that allow simplifying complex expressions without loss of generality; they have been used in econometrics [9], finance [31], and labor economics [44], as well as in general microeconomic [48] and macroeconomic models [20].

For $0 \leq y \ll 1$ and $\alpha \neq 0$, we repeatedly use the truncated Taylor expansions (Sprecher [42])

$$(1+y)^{\alpha} \approx 1 + \alpha y + \alpha(\alpha-1)\frac{y^2}{2} + \alpha(\alpha-1)(\alpha-2)\frac{y^3}{6}, \tag{20}$$

$$\ln(1+y) \approx y - \frac{y^2}{2}, \tag{21}$$

to get, after keeping only first-order terms (here λ, $\omega_1(x)$, and $\Omega(x)$ are all «1),

$$V_{01}(x) - V_0(x) \approx -a\Omega(x) = -a\frac{\rho C}{x}, \tag{22}$$

$$V_1(x) - V_{01}(x) \approx a\frac{\omega_0(x) - \omega_1(x)}{2}. \tag{23}$$

As expected, $V_{01}(x) - V_0(x) < 0$ (project costs are incurred without benefits) while $V_1(x) - V_{01}(x) > 0$ (at the end of the project, congestion drops). We also see that λ does not appear in the first-order expansions (22) and (23), so small values of λ have little impact on x^*.

In the rest of the paper, we adopt the following functional form for $\omega_i(x)$:

$$\omega_i(x) = \gamma_i x^{\delta}. \tag{24}$$

with i=0 before or during, and i=1 after construction; $0<\gamma_1<\gamma_0$; and $\delta>1$. Introducing Equation (24) in Equation (23) leads to

$$V_1(x) - V_{01}(x) \approx a\frac{\gamma_0 - \gamma_1}{2}x^{\delta}. \tag{25}$$

Let us now analyze the deterministic case to benchmark the impact of uncertainty.

III. THE DETERMINISTIC CASE

Let us suppose that the city population increases at a constant rate μ, i.e. Equation (1) is

$$dX_t = \mu X_t dt, \tag{26}$$

until $X_t = \overline{x}$, and once X_t reaches $\overline{x}$, it keeps this value forever. So, if $X_0 = x < \overline{x}$,

$$X_t = \begin{cases} xe^{\mu t}, & \text{if } t \le \dfrac{1}{\mu}\ln\left(\dfrac{\overline{x}}{x}\right), \\ \overline{x}, & \text{otherwise.} \end{cases} \tag{27}$$

Then we have:

Proposition 1. Suppose the impact of the upper population barrier can be neglected. Then, in order to increase welfare, a congestion-relief investment must satisfy the rule of thumb

$$\frac{e^{-\rho\Delta}}{\rho}\frac{\gamma_0 - \gamma_1}{2}\left(x_d^* e^{\mu\Delta}\right)^{\delta+1} \ge C, \tag{28}$$

where the deterministic threshold x_d^* is

$$x_d^* = \left[\frac{2\rho C \exp([\rho - (\delta+1)\mu]\Delta)}{\gamma_0 - \gamma_1} \right]^{\frac{1}{\delta+1}}, \tag{29}$$

with the requirement $\rho - \mu(\delta+1) > 0$ to guarantee that the present value of utility gains is finite. Equation (28) is also approximately equivalent to the first order Taylor expansion of the prescription of a standard benefit-cost analysis (BCA), which requires the present value of the reduction in the total flow of congestion costs to exceed the present value of project costs.

Proof. Let x denote the city population at time 0. The present value of net utility gains from the project, $P(x)$, is given by Equation (17) without the expectation operators. Combine the Taylor approximation (22) with X_t (Equation (27)) to get that the first integral of $P(x)$ equals $-aC$.

The second integral of $P(x)$ is $I_2 \equiv a\frac{\gamma_0 - \gamma_1}{2}\int_{\Delta}^{+\infty} X_t^{\delta+1} e^{-\rho t} dt$. Let T be the time it takes the city population to reach $\overline{x}$ (so $\overline{x} = xe^{\mu T}$). To isolate the impact of $\overline{x}$, rewrite $\int_{\Delta}^{+\infty} X_t^{\delta+1} e^{-\rho t} dt$ as $\int_{\Delta}^{+\infty} \left(xe^{\mu t}\right)^{\delta+1} e^{-\rho t} dt + \int_{T}^{+\infty} \left[\overline{x}^{\delta+1} - \left(xe^{\mu t}\right)^{\delta+1}\right] e^{-\rho t} dt$; the first integral assumes the urban population grows forever, and the second one is a correction for the population barrier $\overline{x}$. Simple calculations show that $\int_{\Delta}^{+\infty} \left(xe^{\mu t}\right)^{\delta+1} e^{-\rho t} dt = e^{-[\rho-(\delta+1)\mu]\Delta} \frac{x^{\delta+1}}{\rho - (\delta+1)\mu}$, which requires $\rho - (\delta+1)\mu > 0$ to be well defined; moreover, the correction term is $\int_{T}^{+\infty} \left[\overline{x}^{\delta+1} - \left(xe^{\mu t}\right)^{\delta+1}\right] e^{-\rho t} dt = -e^{-[\rho-(\delta+1)\mu]T} \frac{\mu(\delta+1)}{\rho} \frac{x^{\delta+1}}{\rho - (\delta+1)\mu}$, so it can be neglected provided $\frac{\mu(\delta+1)}{\rho} \ll e^{[\rho-(\delta+1)\mu](T-\Delta)}$, which we assume. As a result,

$$P(x) = -aC + a\frac{\gamma_0 - \gamma_1}{2}\frac{e^{-[\rho-(\delta+1)\mu]\Delta}}{\rho-(\delta+1)\mu}x^{\delta+1}. \tag{30}$$

Since $\sigma(.)$=0 in Equation (13), the option term is $\varphi_d(x) = F_d x^{\rho\mu^{-1}}$, where F_d is a constant to find jointly with x_d^*. Insert these results in the first-order necessary condition (19) to get Equation (29). To find inequality (28), isolate C on the right side and replace "=" with "≥".

To make the link with the prescriptions of a BCA, we first calculate the total flow of congestion costs with the project when the total city population is x, denoted by $\Phi_1(x) \equiv \int_0^1 \omega_1(x)h\frac{\partial N_1(x,h)}{\partial h}dh$. $\frac{\partial N_1(x,h)}{\partial h}$ is the number of city residents in the two parallel strips of length B and thickness dh at distance h from the CBD. Proceeding as for the derivation of $\frac{\partial N_0(x,h)}{\partial h}$ (see Equation (7)), we obtain

$$\frac{\partial N_1(x,h)}{\partial h} = (1+\nu)x\frac{\left[\lambda+\omega_1(x)\right]\left[1-\Omega(x)-(\lambda+\omega_1(x))h\right]^{\nu}}{\left[1-\Omega(x)\right]^{\nu+1}-\left[1-\Omega(x)-\lambda-\omega_1(x)\right]^{\nu+1}}. \tag{31}$$

After calculating $\Phi_1(x)$, a first-order Taylor expansion using Equation (20) gives

$$\Phi_1(x) \approx \frac{x\omega_1(x)}{2} = \frac{\gamma_1}{2}x^{\delta+1}. \tag{32}$$

To find the flow of congestion costs before (i="0") or during the project (i="01"), $\Phi_i(x) \equiv \int_0^H \omega_0(x)h\frac{\partial N_i(x,h)}{\partial h}dh$, replace $\omega_1(x)$ with $\omega_0(x)$ in Equation (32). A BCA recommends going ahead with the project only if $\int_0^{+\infty}\Phi_0(X_t)e^{-\rho t}dt - \int_0^{\Delta}\Phi_{01}(X_t)e^{-\rho t}dt - \int_{\Delta}^{+\infty}\Phi_1(X_t)e^{-\rho t}dt \geq C$, i.e., if the present value of the reduction in congestion costs exceeds C. Introducing first-order Taylor expansions of $\Phi_0(x)$, $\Phi_{01}(x)$, and $\Phi_1(x)$ into this expression also gives Equation (28) after neglecting the impact of the upper population barrier, which concludes the proof. □

Proposition 1 states that a BCA is a reasonable *practical* guide for deciding when to invest in congestion-relief projects that require no urban land. This contrasts with classical results in urban economics, although classical results are mixed and rely on different assumptions. In their analysis of roadway enlargement in a long, narrow city when traffic congestion is unpriced, Solow and Vickrey [39] found that following a BCA leads to over-investing in roads. In the context of a monocentric circular area, Kanemoto [22] obtained a similar result near the CBD, but the reverse at the city edge if the price elasticity of compensated housing demand is less than one. Kanemoto [21] also showed that following a BCA would lead to under-investing in roadways in a von Thünen framework.

IV. THE STOCHASTIC CASE

We now suppose that the city population follows the geometric Brownian motion (GBM)

$$dX_t = \mu X_t dt + \sigma X_t dw. \tag{33}$$

$\mu>0$ is the population growth rate and $\sigma>0$ is the volatility parameter. X_t thus varies stochastically around an exponential trend.

To derive an explicit solution for x^* under uncertainty, we also assume x^* to be far enough from the population barriers that their impact is discounted away so we can ignore them. This assumption requires μ, σ and Δ to be small enough, and ρ to be large enough; in particular, we require $\mu<\rho$. Unless σ is small, this assumption may not be satisfied by large infrastructure projects serving a volatile population. It may, however, apply well to projects such as equipping some vehicles with the ability of exchanging information to bypass congestion (intelligent vehicle systems). Then:

Proposition 2. Under the assumptions above, the population threshold for investing in congestion relief is

$$x^* = \left[\frac{\theta\kappa}{\theta-\delta-1}\frac{2\rho C\exp(\rho\kappa\Delta)}{\gamma_0-\gamma_1}\right]^{\frac{1}{\delta+1}}, \tag{34}$$

where θ is the positive root of the quadratic function

$$f(z) \equiv 0.5\sigma^2 z^2 + \left(\mu - 0.5\sigma^2\right)z - \rho = 0, \tag{35}$$

so that

$$\theta = \frac{0.5\sigma^2 - \mu + \sqrt{(0.5\sigma^2-\mu)^2 + 2\rho\sigma^2}}{\sigma^2}. \tag{36}$$

A little bit of algebra shows that $\mu<\rho$ (see A3) implies $\theta>1$. Moreover, κ is given by

$$\kappa = 1 - \frac{\delta+1}{\rho}[\mu + \frac{\sigma^2}{2}\delta]. \tag{37}$$

Proof: Insert the Taylor expansions (22) and (23) into the expression of $P(x)$ (Equation (17)). A simple calculation shows that the first integral of $P(x)$ equals $-aC$.

The second integral of $P(x)$ is $I_2 \equiv a\frac{\gamma_0-\gamma_1}{2}\int_{\Delta}^{+\infty} E\left\{X_t^{\delta+1}\middle|X_0 = x\right\}e^{-\rho t}dt$. Since X_t

follows a GBM (see Equation (33)) and we ignore population barriers, we know from Dixit and Pindyck ([12], p. 82) that $E\left\{X_t^{\delta+1}\middle|X_0 = x\right\} = x^{\delta+1}e^{(\delta+1)[\mu+\frac{\sigma^2}{2}\delta]t}$. Insert this result in I_2 and integrate to find $I_2 = a\frac{\gamma_0 - \gamma_1}{2}\frac{e^{-\rho\kappa\Delta}}{\rho\kappa}x^{\delta+1}$. It follows that

$$P(x) = -aC + a\frac{\gamma_0 - \gamma_1}{2}\frac{e^{-\rho\kappa\Delta}}{\rho\kappa}x^{\delta+1}. \tag{38}$$

It is important to note, however, that I_2 is well defined provided κ>0, which requires

$$\sigma^2 < \frac{2}{\delta}\left(\frac{\rho}{\delta+1} - \mu\right). \tag{39}$$

To derive the option term, solve analytically Equations (13)-(14) with $\underline{x}$>0 to find $\varphi(x) = F\left\{x^\theta - \frac{\theta}{\theta_2}\underline{x}^{\theta-\theta_2}x^{\theta^2}\right\}$, where: $\theta_2 = \frac{0.5\sigma^2 - \mu - \sqrt{(0.5\sigma^2 - \mu)^2 + 2\rho\sigma^2}}{\sigma^2} < 0$; $\theta > 1$ is defined by Equation (36) (θ and θ_2 are the roots of $f(z)$); and F is a constant to find jointly with x^*. The term in $\underline{x}^{\theta-\theta_2}$ is the contribution of the lower population barrier, which we ignore, so

$$\varphi(x) = Fx^\theta. \tag{40}$$

Insert Equations (38) and (40) in the first order condition (19) and solve to get Equation (34). □

To assess the impact of population uncertainty on the timing of a congestion-relief investment, let us combine x^* with x_d^* (see Equation (29)) to get:

$$x^* = x_d^*\left(\frac{\theta\kappa}{\theta - \delta - 1}\right)^{\frac{1}{\delta+1}}\exp\left(-\frac{\sigma^2}{2}\delta\Delta\right). \tag{41}$$

We then have:

Proposition 3. If Δ=0, x^* increases with the population volatility σ. When Δ>0, for small values of σ (recall constraint (39)), x^* increases with σ provided $\Delta < \frac{1}{\delta\mu}$ and it decreases otherwise.

Proof: See the appendix. □

Proposition 3 highlights the importance of the interplay between the time lag and uncertainty. In this context, what is the consequence of ignoring uncertainty on the timing of congestion relief? In the proof of Proposition 3, we show that $\frac{\theta\kappa}{\theta-\delta-1}$ increases with σ, and from Equation (A.2), $\lim_{\sigma\to 0+}\frac{\theta\kappa}{\theta-\delta-1}=1$, so $\frac{\theta\kappa}{\theta-\delta-1}>1$ for σ>0. Δ does not appear in $\frac{\theta\kappa}{\theta-\delta-1}$, so we infer from Equation (41) that $\frac{x^*}{x_d^*}>1$ for small values of Δ, but $\frac{x^*}{x_d^*}<1$ for Δ large enough, because of the term $\exp\left(-\frac{\sigma^2}{2}\delta\Delta\right)$. Making this explicit leads to:

Proposition 4. When our assumptions hold, ignoring uncertainty leads to investing prematurely in congestion relief if $\Delta < \frac{2}{\sigma^2\delta(\delta+1)}\ln\left(\frac{\theta\kappa}{\theta-\delta-1}\right)$. If the project duration Δ is longer, however, ignoring uncertainty leads to investing too late.

These findings contrast with McDonald and Siegel's results [26] (also see Chapter 5 in [12]) for their most basic model. They show that increasing uncertainty always delays the decision to invest a fixed amount to purchase an asset whose value follows a GBM. Here, increasing uncertainty augments the gains from congestion relief, but also the flow of congestion costs during project implementation; delaying the project further discounts project gains. If uncertainty is high enough and the project implementation long enough, it becomes optimal to invest in congestion relief earlier rather than later, when uncertainty increases. Just as in Bar-Ilan and Strange's [3] study of land conversion, uncertainty has an ambiguous impact here because it affects both the benefits and the costs of investing in congestion relief.

To further contrast the deterministic and the stochastic cases, rewrite Equation (34) as

$$a\frac{\gamma_0-\gamma_1}{2}\frac{e^{-\rho\kappa\Delta}}{\rho\kappa}(x^*)^{\delta+1}=\frac{\theta}{\theta-\delta-1}aC, \tag{42}$$

and compare it to the equivalent expression for the deterministic case, obtained by reorganizing Equation (29) (a, the utility elasticity of the numeraire appears on purpose):

$$a\frac{\gamma_0-\gamma_1}{2}\frac{e^{-[\rho-(\delta+1)\mu]\Delta}}{\rho-(\delta+1)\mu}(x_d^*)^{\delta+1}=\frac{\rho}{\rho-(\delta+1)\mu}aC. \tag{43}$$

Now jointly consider Equations (42) and (43). Their left side is the first-order Taylor expansion of the present value of expected utility gains from congestion reduction (recall the derivations of Equations (30) and (38)); on their right side, aC is the first order Taylor expansion of the present value of utility losses from the project. Now note that setting μ to 0 in Equation (43) makes $\frac{\rho}{\rho-(\delta+1)\mu}$ equal to 1, so $\frac{\rho}{\rho-(\delta+1)\mu}$ is the multiplier by which gains have to exceed losses at the optimum, when the population grows at a constant rate μ. Following

McDonald and Siegel [26], we therefore interpret the ratio of $\frac{\theta}{\theta-\delta-1}$ to $\frac{\rho}{\rho-(\delta+1)\mu}$ as the wedge driven by uncertainty between utility gains and losses. Taking the derivative with respect to σ of this ratio and using Equation (A.1) shows that this wedge increases with uncertainty (but so does $\frac{e^{-\rho\kappa\Delta}}{\rho\kappa}$, which explains the possible non-monotonic behavior of x^*).

To illustrate the joint impacts of σ and Δ on the difference between x^* and x_d^*, let us graph $\frac{x^* - x_d^*}{x_d^*}100\%$, the percentage change in the stochastic population threshold x^* compared to the deterministic population threshold x_d^*, against the population volatility σ, for different values of Δ, the time interval needed to implement the project. We set the population growth rate μ to 0.5% per year, the congestion exponent δ to 4, and the social discount rate ρ to 7 % per year. From Equation (39), we know that σ needs to be smaller than 0.067 for the present value of expected congestion costs to be finite. Results were generated via Excel on a PC; they are presented in Figure 2. In agreement with Proposition 4, $\frac{x^* - x_d^*}{x_d^*}100\%$ increases with σ for smaller values of Δ (5 years), increases and then decreases with σ when Δ takes on intermediate values (10 or 20 years), and decreases with σ for $\Delta \geq \frac{1}{\mu\delta}$ (=50 here).

V. CONCLUSIONS

This paper analyzes the impacts of uncertainty and irreversibility on the timing of an urban congestion-relief investment that requires time to bear fruit but little or no urban land. To date, the urban economics literature typically analyzes deterministic models with a static population; only a handful of papers allow for deterministic urban growth.

We obtain two important results. When the urban population grows at a constant rate, we derive a utility-based rule of thumb and show that it is equivalent to a standard benefit-cost analysis. Under uncertainty, we derive an explicit population threshold for investing in congestion relief when the urban population follows a geometric Brownian motion. We find that following a standard benefit-cost ratio may lead to investing prematurely or too late, depending on the level of uncertainty and the time necessary to implement the project. This is qualitatively similar to the result obtained numerically by Bar-Ilan and Strange [3] in their study of land conversion under uncertainty. Our findings contrast with the basic real options investment model (see Chapter 5 in [12]) because uncertainty has an ambiguous impact in our framework: increasing uncertainty augments the gains from congestion relief, but also the flow of congestion costs during project implementation; delaying the project further discounts project gains. If uncertainty is high enough and the project implementation long enough, it becomes optimal to invest earlier in congestion relief as uncertainty increases.

Future work could assess the practical impact of our results on the timing of actual congestion-relief investments with a focus on large infrastructure projects in cities with volatile populations. Other interesting expansions include exploring the value of buying land for future infrastructure needs (land banking); analyzing the timing of infrastructure investments in expanding urban areas; considering the impact of federal subsidies; or jointly analyzing timing and capacity choice. In addition, our framework could be expanded to study other externalities linked to random population fluctuations.

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APPENDIX

In this appendix, we prove Proposition 3. Let us first suppose that Δ=0, so Equation (41) becomes $x^* = x_d^* \left[\frac{\theta\kappa}{\theta - \delta - 1} \right]^{\frac{1}{\delta+1}}$. To show that $\frac{dx^*}{d\sigma} > 0$, it is useful to introduce $g \equiv \left[\frac{(\theta-1)\kappa}{\theta - \delta - 1} \right]^{\frac{1}{\delta}}$ and prove that $\frac{dg}{d\sigma} > 0$. We have $\frac{dg}{d\sigma} = \frac{\partial g}{\partial \kappa}\frac{d\kappa}{d\sigma} + \frac{\partial g}{\partial \theta}\frac{d\theta}{d\sigma}$ with $\frac{\partial g}{\partial \kappa} = \frac{g}{\delta\kappa}$, $\frac{d\kappa}{d\sigma} = -\frac{\delta(\delta+1)}{\rho}\sigma$, $\frac{\partial g}{\partial \theta} = \frac{-g}{(\theta-1)(\theta-[\delta+1])}$, and

$$\frac{d\theta}{d\sigma} = \frac{-\sigma\theta^2(\theta-1)}{0.5\sigma^2\theta^2 + \rho}. \tag{A.1}$$

To derive this result, apply the implicit function theorem to $f(\theta)$ (see Equation (35)) as a function of θ and σ: the relationship $\frac{d\theta}{d\sigma} = -\left(\frac{\partial f}{\partial \sigma}\right)\left(\frac{\partial f}{\partial \theta}\right)^{-1}$ leads to Equation (A.1). After some algebra, combining expressions of $\frac{\partial g}{\partial \kappa}$, $\frac{d\kappa}{d\sigma}$, $\frac{\partial g}{\partial \theta}$, and $\frac{d\theta}{d\sigma}$ simplifies to $\frac{dg}{d\sigma} = \frac{\theta - \delta - 1}{\kappa[0.5\sigma^2\theta^2 + \rho]} g$. To find the sign of $\theta - \delta - 1$, recall from Proposition 2 that κ>0, or equivalently $\frac{\sigma^2}{2}(\delta+1)^2 + (\mu - \frac{\sigma^2}{2})(\delta+1) - \rho < 0$. Necessarily $\delta + 1 < \theta$, then, since $f(z)$<0 only between its roots (θ is the largest root of $f(z)$). Hence $\frac{dg}{d\sigma} > 0$. Next, calculate $\frac{dx^*}{d\sigma}$ from $x^* \equiv x_d^* g^{\frac{\delta}{\delta+1}} \left[\frac{\theta}{\theta - 1} \right]^{\frac{1}{\delta+1}}$; as $\frac{d\theta}{d\sigma} < 0$ and $\frac{dg}{d\sigma} > 0$, $\frac{dx^*}{d\sigma} > 0$ because $\frac{dx^*}{d\sigma}$ is the sum of positive terms. This shows the first part of Proposition 3.

Now consider the case $\Delta>0$ for small values of σ only because of the constraint (39). In Equation (41), the second term on the right side of x^* increases with σ but the third one decreases with σ; x_d^* does not depend on σ. A repeated use of the Taylor expansion (20) for θ and κ yields

$$\left[\frac{\theta\kappa}{\theta-\delta-1}\right]^{\frac{1}{\delta+1}} \approx 1+\frac{\sigma^2}{2\mu}, \tag{A.2}$$

and from the Taylor expansion $e^y \approx 1+y$ with y small, we obtain $\exp\left(-\frac{\sigma^2}{2}\delta\Delta\right) \approx 1-\frac{\sigma^2}{2}\delta\,\Delta$.

When we multiply this expression with (A.2) and discard terms smaller than σ^2, we get

$$x^* \approx [1+\frac{\sigma^2}{2\mu}(1-\mu\delta\Delta)]x_d^*, \tag{A.3}$$

so x^* increase with σ if $\Delta<\frac{1}{\delta\mu}$, and it decreases otherwise; in addition, $\lim_{\sigma\to 0} x^* = x_d^*$.

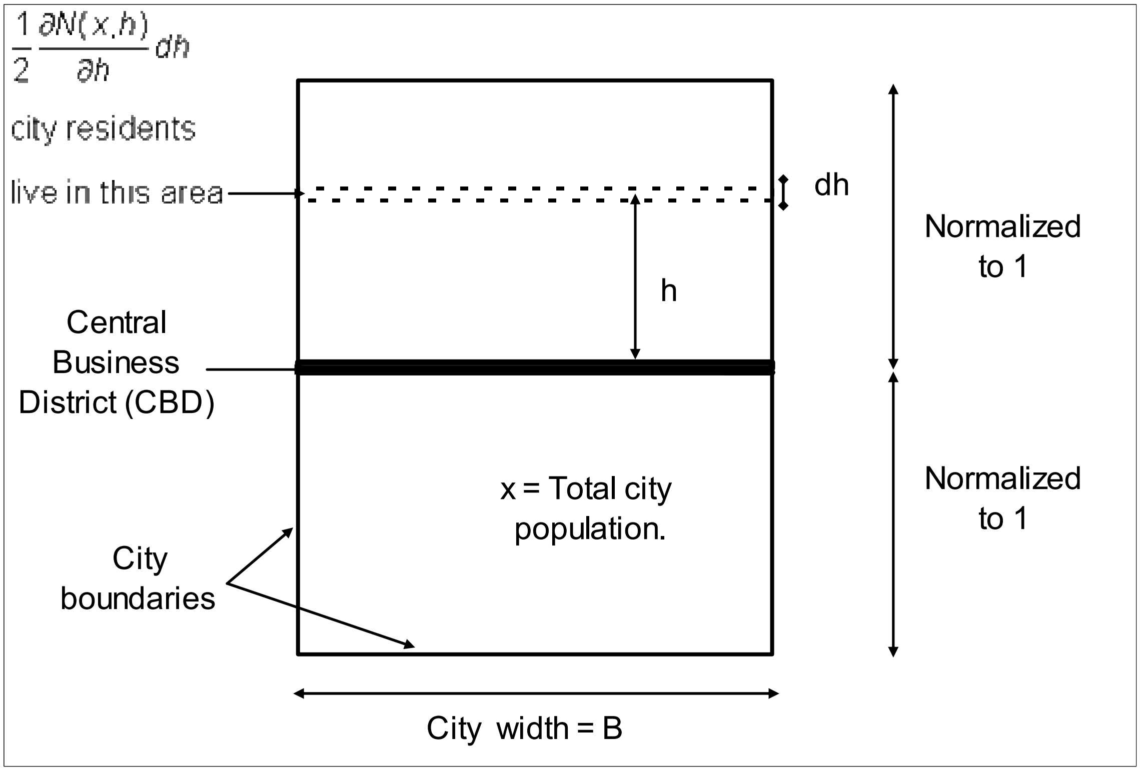

Figure 1: City map.

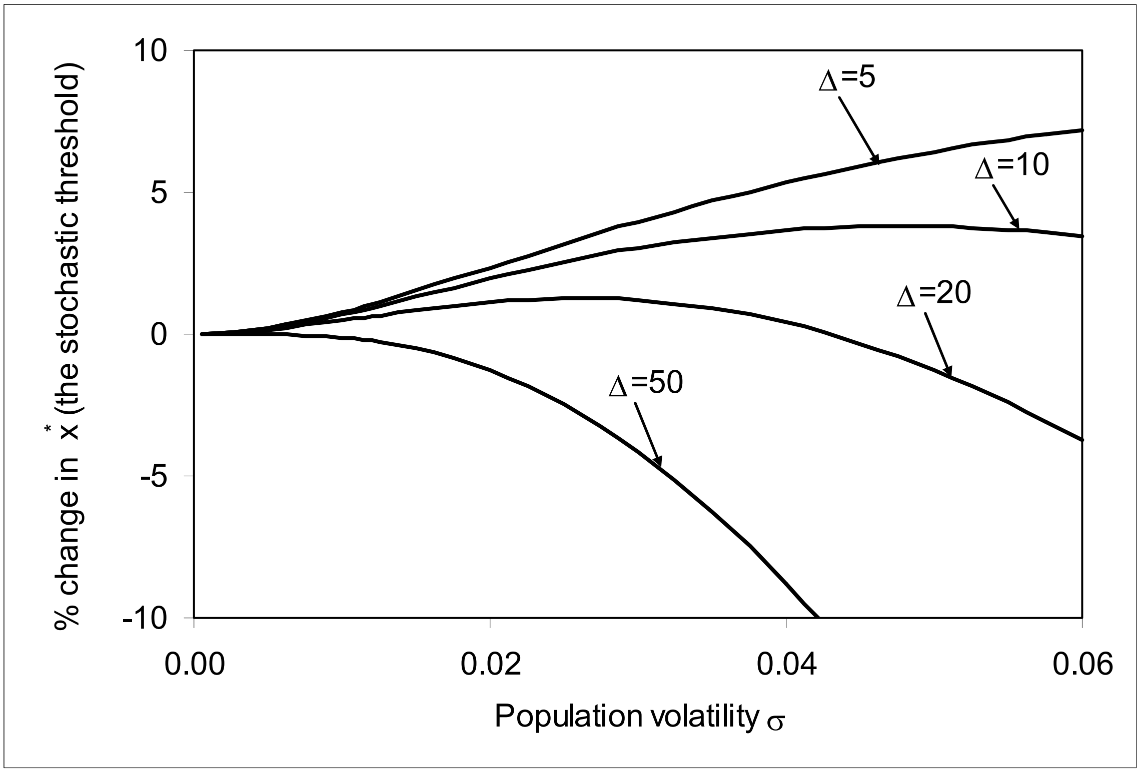

Figure 2: Relative change in the stochastic population threshold versus population volatility.

Notes. The relative change in the stochastic population threshold x^* is calculated as $\frac{x^* - x_d^*}{x_d^*} 100\%$, with $\frac{x^*}{x_d^*} = \left(\frac{\theta\kappa}{\theta - \delta - 1}\right)^{\frac{1}{\delta+1}} \exp\left(-\frac{\sigma^2}{2}\delta\Delta\right)$ from Equation (41); x_d^* is the deterministic population threshold for investing in congestion relief. See Equation (36) for θ.

These results were generated with an annual population growth rate μ of 0.5% (refer to Equation (33)), a congestion relief exponent δ (defined in Equation (24)) equal to 4 and an annual social discount rate ρ of 7%. Δ is the time (in years) needed to implement the project.

Table 1: Definitions of Key Parameters.

Name	Description
a, 1-a	Utility elasticities of the numeraire and of land .
B	City width (see Figure 1).
C	Total cost of the project, including future maintenance and operation.
δ	Congestion coefficient (see Equation (24)).
Δ	Time necessary to implement the project.
g	Fraction of the flow of individual income spent on the numeraire.
h	Aerial distance from the CBD to a point in the city (see Figure 1); $0\leq h \leq 1$.
γ_i	Congestion multiplicative factor (see Equation (24)).
λ	Fraction of the flow of individual income spent on transportation costs.
L	Land area consumed by a city resident (it depends on h and x).
μ	Population growth rate in the GBM model (see Equation (33)).
ν	$\nu \equiv a / (1 - a)$ is a ratio of elasticities.
ρ	Social discount rate.
σ	Population volatility in the GBM model (see Equation (33)).
$\omega_1(x)$	Fraction of the flow of individual income spent on congestion costs for a resident living at the city edge (h=1).
$\Omega(x) \equiv \dfrac{\rho C}{x}$	Fraction of the flow of total city income spent on project payments.
$\underline{x}$, $\overline{x}$	Lower and upper bounds on the city population.
x^*	Stochastic population threshold for investing in congestion relief.
x_d^*	Deterministic population threshold for investing in congestion relief when the city population grows at a constant rate μ.

Notes. Both the flow of individual income and the aerial distance from the CBD to the city edge (see Figure 1) are normalized to be 1.