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**The Dynamic System of the
Traffic Assignment Problem:
Part ■. Incorporating Traffic Dynamics**

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Abstract

In the two preceding papers, we established the so-called J-system as *the* dynamic system of the traffic assignment problem for static case and developed corresponding numerical solution methods. In this paper, we present J-system formulation of the dynamic traffic assignment problem. Here we first extend the definition of J-functions of First-In-First-Out violation for a dynamic traffic network. Then we propose both discrete and continuous J-systems, whose steady states are defined as (dynamic) J-equilibria (JE) and stable steady states as (dynamic) user equilibria (UE). Then we present finite difference method and perturbation-based method intrinsic to J-system for solving JE and UE. For a simple network we show that the finite difference method gives convergent solutions measured by J-index of convergence and that the perturbation-based method yields UE solutions. This finishes our initial discussions of the dynamic system of the traffic assignment problem. We expect these studies to have fundamental and profound impacts on studying transportation systems and other social systems with user equilibria.

Keywords: Dynamic traffic assignment problems, J-functions of First-In-First-Out violation, J-equilibria, J-system, User equilibria, Finite difference method, Perturbation-based

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method, J-index of convergence

1 Introduction

In (Jin, 2005b), we defined J-functions of First-In-First-Out (FIFO) violation and a so-called J-system of the static traffic assignment problem (TAP), an autonomous dynamic system of changing rules for all path flows. Without traffic dynamics, we showed that J-system is *the* dynamic system of TAP in the sense that the BMW objective function (Beckmann, McGuire, and Winsten, 1956) is its total energy and user equilibria (UE) are its only globally stable steady states, among many so-called J-equilibria (JE). Based on this new definition of UE, we further developed the finite difference method and perturbation-based method, which are intrinsic to J-system, for solving JE and searching UE (Jin, 2005a).

As pointed out in (Jin, 2005b), the definition of J-functions of FIFO violation was inspired by the measurements of FIFO violation defined in (Jin and Jayakrishnan, 2005). Since traffic dynamics was concerned in (Jin and Jayakrishnan, 2005), it is not surprising that we can also define J-functions of FIFO violation among dynamic path flows and therefore obtain J-system of dynamic TAP. Here we directly define JE and UE as the steady states and stable steady states of J-system respectively. For traditional definitions and formulations of dynamic TAP, please refer to (Lo, 1999) and the references therein. Based on this definition, we can also apply the finite difference method and perturbation-based method intrinsic to J-system for computing JE and UE. The feasibility and validity of this new formulation was demonstrated by an example.

In this paper, all quantities are time-dependent, i.e., dynamic. Since the concepts and definitions are all similar to those in static case, we do not specifically include “dynamic” in them and keep the same terminologies as before. For example, UE is really dynamic UE. Note that this formulation is purely path-based, and we assume no link performance functions, which unlikely exist in dynamic traffic with capacity constraints and link interactions (Daganzo, 1995b). In this study we adopt the network notation system given in **Table**

1. The following remarks are made in order. First, flow-rates are derivatives of flows, and their units are respectively number of vehicle per unit time and number of vehicles. Second, all variables depend on the decision variable τ , although not explicitly indicated. Third, we do not include link flow, link travel time, or indicator variable since the formulation is path-based. In addition, we have the following basic relationships:

$$p_{rs}(r, t) = \sum_k f_k^{rs}(r, t), \quad p_{rs}(s, t) = \sum_k f_k^{rs}(s, t), \quad \forall r, s \quad (1a)$$

$$f_k^{rs}(r, t) \geq 0, \quad f_k^{rs}(s, t) \geq 0, \quad \forall k, r, s; \quad (1b)$$

where p_{rs} and f_k^{rs} are cumulative arrival curves at origins and destinations (Moskowitz and Newman, 1963; Newell, 1993), and

$$q_{rs}(r, t) = \sum_k g_k^{rs}(r, t), \quad q_{rs}(s, t) = \sum_k g_k^{rs}(s, t), \quad \forall r, s \quad (1c)$$

$$g_k^{rs}(r, t) \geq 0, \quad g_k^{rs}(s, t) \geq 0, \quad \forall k, r, s. \quad (1d)$$

We also use abbreviations in **Table 2**.

The rest of the paper is organized as follows. In Section 2, we define J-functions of FIFO violation, derive J-system of the dynamic TAP, and present a definition of UE. In Section 3, we present algorithms for finding JE and UE. In Section 4, we study a simple example. Finally, we make some discussions in Section 5.

2 J-system of the dynamic traffic assignment problem

2.1 J-functions of FIFO violation and J-system

We consider two paths, 1 and 2, connecting one O-D pair $r-s$ in a network. The cumulative arrival curves at the origin and the destination for O-D flow and path flows are shown in Figure 1. Here we assume that there is no FIFO violation among vehicles on the same path. Therefore, cumulative flow can be used as the identity of a vehicle (Jin and Zhang, 2004). As shown in the figure, vehicles $f_1^{rs}(r, t)$ and $f_2^{rs}(r, t)$ arriving at the origin at the same time t arrive at the destination at different times. Such FIFO violation can also be observed for

other vehicles. That is, there is FIFO violation among the two paths. From t to $t + \Delta t$, the total travel time on path 1 for vehicles from $f_1^{rs}(r, t)$ to $f_1^{rs}(r, t + \Delta t)$ is the area bounded by the two thick solid curves; and similarly we can find the total travel time on path 2. In one approach, the total travel time for the O-D pair can be considered as the area bounded by the two dashed curves. As we know, if there is no FIFO violation among this time interval, these three areas should be proportional to the number of vehicles: $\Delta f_1^{rs}(r, t)$, $\Delta f_2^{rs}(r, t)$, and $\Delta p_{rs}(r, t)$, where $\Delta(\cdot) = (\cdot)(r, t + \Delta t) - (\cdot)(t)$. Since these areas can be considered as integral of travel times, we can define J-function of FIFO violation for path k connecting O-D pair $r - s$ during a time interval $[t, t + \Delta t]$ as

$$J_k^{rs}(r, t)\Delta t = \frac{\Delta p_{rs} \int_{f_k^{rs}(r, t)}^{f_k^{rs}(r, t + \Delta t)} c_k^{rs} df_k^{rs} - \Delta f_k^{rs} \int_{p_{rs}(r, t)}^{p_{rs}(r, t + \Delta t)} v_{rs} dp_{rs}}{\Delta t}, \quad \forall k, r, s, \quad (2a)$$

where v_{rs} is travel time for O-D flow. In another approach, we define the total travel time of O-D flows as the sum of path travel times, that is, we can have another definition of J-function of FIFO violation

$$J_k^{rs}(r, t)\Delta t = \frac{\Delta p_{rs} \int_{f_k^{rs}(r, t)}^{f_k^{rs}(r, t + \Delta t)} c_k^{rs} df_k^{rs} - \Delta f_k^{rs} \sum_j \int_{f_j^{rs}(r, t)}^{f_j^{rs}(r, t + \Delta t)} c_j^{rs} df_j^{rs}}{\Delta t}, \quad \forall k, r, s \quad (2b)$$

Although these two quantities are usually different, if the time interval is small enough, we expect them to be the same. Note that $J_k^{rs}(r, t)\Delta t$ can be considered as total FIFO violation over Δt .

Then we define the following J-system:

$$-\frac{d\Delta f_k^{rs}(r, t)}{d\tau} = J_k^{rs}(r, t)\Delta t, \quad \forall k, r, s \quad (3)$$

which is in finite difference form in t . If dividing both sides of (3) by Δt and letting $\Delta t \rightarrow 0$, we have the following continuous system

$$-\frac{dg_k^{rs}(r, t)}{d\tau} = J_k^{rs}(r, t), \quad \forall k, r, s \quad (4)$$

where, corresponding to (2a) and (2b), J-functions are

$$J_k^{rs}(r, t) = q_{rs}(r, t)g_k^{rs}(r, t)(c_k^{rs}(r, t) - v_{rs}(r, t)), \quad \forall k, r, s \quad (5a)$$

or

$$\begin{aligned}
J_k^{rs}(r, t) &= q_{rs}(r, t)g_k^{rs}(r, t)c_k^{rs}(r, t) - g_k^{rs}(r, t) \sum_j g_j^{rs}(r, t)c_j^{rs}(r, t) \\
&= g_k^{rs}(r, t) \sum_j g_j^{rs}(r, t)(c_k^{rs}(r, t) - c_j^{rs}(r, t)), \forall k, r, s
\end{aligned} \tag{5b}$$

Note that here travel times include waiting times, since we consider arrival not departure flows at origins.

2.2 Dynamic J-equilibria and user equilibria

Definition 2.1 (Definition of dynamic J-equilibria (JE)) *If $\Delta f_k^{rs}(r, n\Delta t)$ for any n is a JE of J-system (3) over an interval $[n\Delta t, (n+1)\Delta t]$ for any n , then we call the series $f_k^{rs}(r, n\Delta t)$ as a discrete JE. If for any t , $g_k^{rs}(r, t)$ ($\forall k, r, s$) is a JE of (4), then we call functions $g_k^{rs}(r, t)$ as a continuous JE.*

Remarks: Note that with (2b), we can obtain trivial discrete JE by assigning all vehicles to one route during *each* time interval. However, these may not be discrete JE with (2a), for which the trivial discrete JE is obtained by assigning all vehicles to one route during *all* time intervals. This difference does not exist for continuous JE, since, at JE, both (5a) and (5b) are equivalent to that all non-empty path travel times are the same and equal O-D travel time. We expect that discrete JE converge to continuous JE with $\Delta t \rightarrow 0$.

Definition 2.2 (Definition of dynamic user equilibria (UE)) *Discrete series $f_k^{rs}(r, n\Delta t)$ ($\forall k, r, s$) is a discrete UE iff it is a discrete JE and globally stable in all time intervals. Functions $g_k^{rs}(r, t)$ ($\forall k, r, s$) are continuous UE iff it is a continuous JE and globally stable.*

Remarks: We also expect that discrete UE converge to continuous UE with $\Delta t \rightarrow 0$. However, the existence or uniqueness of UE is not discussed here.

3 Computation

In this section, we describe computational methods for finding JE and UE defined in Section 2. These methods are based on discretization of J-system in time. Here we consider an assignment time interval $[0, T_0]$ with time instants $t_n = n\Delta t$, where $n = 0, 1, 2, \dots, N$ and $N = T_0/\Delta t$. We assume traffic demand after this time interval as 0. We consider traffic simulation time interval as $[0, T]$, where $T > T_0$ should be long enough to ensure all vehicles arrive at their destinations during this time interval.

3.1 Computation of total travel time

Given an initial guess of $f_k^{rs}(r, t_n)$ ($\forall k, r, s$), boundary conditions at destinations, and initial traffic conditions, we can use any traffic simulator to obtain solutions of arrival curves at destinations $f_k^{rs}(s, t)$ at a number of time instants. Then we can find the time instant t_i when $f_k^{rs}(s, t_i) = f_k^{rs}(r, t_n)$. If we use the same time steps in traffic simulation, then $f_k^{rs}(s, t_i)$ closest to $f_k^{rs}(r, t_n)$ is subject to difference, e.g. in the order of Δt in CKW model (Jin and Zhang, 2004). Here we use smaller time steps in traffic simulator as $\Delta t/M$ with $M > 1$. Therefore, in a traffic simulator, we have time instants of $t_i = i\frac{\Delta t}{M}$ where $i = 0, 1, 2, \dots, MNT/T_0$.

To compute the total travel time of vehicles from $f_k^{rs}(r, t_n)$ to $f_k^{rs}(r, t_{n+1})$, we first find time instants t_{m_n} and $t_{m_{n+1}}$ such that $f_k^{rs}(s, t_{m_n})$ to $f_k^{rs}(s, t_{m_{n+1}})$ are the closest to the former two flows respectively. Then we have, as illustrated in Figure 2,

$$\begin{aligned} \int_{f_k^{rs}(r, t_n)}^{f_k^{rs}(r, t_{n+1})} c_k^{rs} df_k^{rs} &\approx (f_k^{rs}(r, t_{n+1}) - f_k^{rs}(r, t_n)) \left(\frac{m_{n+1}}{M} - n - \frac{1}{2} \right) \Delta t \\ &\quad - \sum_{i=m_n+1}^{m_{n+1}} (f_k^{rs}(s, t_i) - f_k^{rs}(s, t_{m_n})) \frac{\Delta t}{M} \\ &\quad + \frac{1}{2} (f_k^{rs}(s, t_{m_{n+1}}) - f_k^{rs}(s, t_{m_n})) \frac{\Delta t}{M}, \end{aligned} \quad (6)$$

whose estimation error is in the order of $\Delta t/M$. O-D travel time, integral in v_{rs} , can also be calculated in this approach. From these travel times, we can then compute J-functions

as in (2a) or (2b).

3.2 Computation of JE and UE

Once obtaining J-functions $J_k^{rs}(r, t_n)\Delta t$ ($\forall k, r, s$), we can solve (3) by a finite difference approximation as follows

$$\Delta f_k^{rs}(r, t_n)|_{\tau+\Delta\tau} = \Delta f_k^{rs}(r, t_n)|_{\tau} - J_k^{rs}(r, t_n)\Delta t\Delta\tau, \quad \forall k, r, s, \quad (7)$$

where $n = 0, 1, \dots, N - 1$, and $f_k^{rs}(r, 0) = 0$ for any τ . Then from these solutions, we can construct a time series of $f_k^{rs}(r, t_n)$ at $\tau + \Delta\tau$. Note that, since computation error can be introduced when computing travel times by using (6), $\Delta f_k^{rs}(r, t_n)|_{\tau+\Delta\tau}$ has to be adjusted to satisfy (1). This algorithm for finding JE is shown in **Table 3**. Here we use the following J-index to measure convergence of solutions

$$\|\mathbf{J}\|_2 = \sqrt{\frac{\sum_{rs} \sum_k \sum_{n=1}^N (J_k^{rs}(r, t_n))^2}{N \sum_{rs} K_{rs}}}, \quad (8)$$

where K_{rs} is the number of initially non-empty paths connecting O-D pair $r - s$. Since for small enough $\Delta\tau$, the finite difference equations are stable and convergent, solutions will converge to JE for big enough τ . Then letting $\Delta t \rightarrow 0$, we can obtain the continuous JE. To save computational time, one can also use the fourth-order Runge-Kutta method (Strogatz, 1994).

Then according to the definition of UE as stable steady states of J-system, we search for UE by perturbing any JE, where all taken paths have the same travel time. We always perturb the whole time series of $f_k^{rs}(r, t_n)$ ($\forall n = 1, \dots, N$), although partially perturbing their values at several time instants might work.

Note that, in this new definition of UE, we do not consider un-taken paths or dynamic shorter paths. This treatment, on one side, is good since it avoids using dynamic shortest paths, which are not trivial to find. On the other hand, this may result in some difficulties in finding a good perturbation direction to get to UE. As a suggestion, when we do not have ideas about better perturbation directions, we may simply choose them randomly.

4 An example

In this section, we study the dynamic TAP of the network shown in Figure 3, where two paths have the same triangular fundamental diagram (Munjal et al., 1971; Newell, 1993). The network is empty initially, and there is no capacity constraint at the destination. We can see that the free-flow travel times on two paths are 1 and 2 respectively. Here we consider a constant arrival flow-rate at the origin, q_0 , during time interval $[0, T_0]$. That is, $q_{rs}(r, t) = q_0$ for $t \in [0, T_0]$ and 0 otherwise.

4.1 Traffic flow models

We use the Lighthill-Whitham-Richards (Lighthill and Whitham, 1955; Richards, 1956) traffic model to analyze traffic dynamics. For example, for link 1, however many vehicles are waiting to enter, the maximum flow-rate is $q_c = 1$. Since there is no bottleneck on the link or the destination, we know that traffic on link 1 is always free flow; i.e., travel time is always 1. Although the link travel time is constant, the waiting time at the origin is time-dependent. If the arrival flow at the destination for path 1 is $f_1(s, t)$, then the departure flow at the origin is $f_1(s, 1 + t)$. Denoting $t_i = i\Delta t/M$, we then have the following equation

$$f_1(s, 1 + t_{i+1}) = f_1(s, 1 + t_i) + \frac{\Delta t}{M} \min\left\{q_c, \frac{f_1(r, t_i) - f_1(s, 1 + t_i)}{\Delta t} M + g_1(r, t_i)\right\}, \quad (9a)$$

where $f_1(s, 1) = 0$, $f_1(r, 0) = 0$, and $\frac{f_1(r, t_i) - f_1(s, 1 + t_i)}{\Delta t} M + g_1(r, t_i)$ is the maximum flow-rate that can be sent from the origin if there is no capacity constraint on this link. This method of computing boundary fluxes is based on the supply-demand method (Daganzo, 1995a; Lebacque, 1996; Jin and Zhang, 2003). Thus, (9a) gives the dynamic model for finding arrival flows at the destination on path 1. Similarly, we can have the following model for path 2:

$$f_2(s, 2 + t_{i+1}) = f_2(s, 2 + t_i) + \frac{\Delta t}{M} \max\left\{q_c, \frac{f_2(r, t_i) - f_2(s, 2 + t_i)}{\Delta t} M + g_2(r, t_i)\right\}, \quad (9b)$$

where $f_2(s, 2) = 0$ and $f_2(r, 0) = 0$.

4.2 Solutions of the dynamic TAP

We choose $T_0 = 1$, $T = 8$, $N = 20$, $M = 10$, $\Delta t = T_0/N$, $q_0 = 5$, $\tau = 160$, and $\Delta\tau = 0.05$.

We consider deterministic initial time series of $g_1(r, t_n) = cq_0$ and $g_2(r, t_n) = (1 - c)q_0$ ($n = 1, \dots, N$). When $c = 0.5$; i.e., both paths have the same flows initially, solutions of cumulative arrival curves of path flows and total flows at the origin and the destination are shown in (4), from which we can clearly see that these solutions satisfy FIFO. At different time instants, the travel times are shown in Figure 5, from which it is confirmed that the solutions are JE. From both figures, we can see that this JE is given by

$$f_1(r, t) = \begin{cases} 5t, & 0 \leq t \leq 0.25; \\ 1.25 + 2.5(t - 0.25), & 0.25 < t \leq 1; \\ 3.125, & t > 1, \end{cases} \quad (10a)$$

$$f_2(r, t) = \begin{cases} 0, & 0 \leq t \leq 0.25; \\ 2.5(t - 0.25), & 0.25 < t \leq 1; \\ 1.875, & t > 1. \end{cases} \quad (10b)$$

With the aforementioned deterministic initial time series with $c = 0.5$, 0.95 , and 0.05 , and another random initial time series, we find that all solutions converge to (10) and J-index of convergence is shown in Figure 6, from which we can see that solutions converge quickly at the beginning stage, then slowly, but exponentially when it is closer to (10). Note that $g_1(r, t_n) = q_0$ ($\forall n = 1, \dots, N$) and $g_2(r, t_n) = q_0$ ($\forall n = 1, \dots, N$) are two trivial JE. Since initial time series with $c = 0.95$ can be considered as a perturbation around $g_1(r, t) = q_0$ and those with $c = 0.05$ a perturbation around $g_2(r, t) = q_0$, we can conclude that JE in (10) is stable and therefore a UE. Especially, convergence with random initial time series further confirms this.

5 Discussions

In this paper, we derived both discrete and continuous J-systems of the dynamic traffic assignment problem and defined corresponding JE and UE, the only globally stable JE. We also presented the finite difference method and the perturbation-based method intrinsic to J-system for finding JE and UE. By obtaining UE of a simple problem, we demonstrated the feasibility and validity of this new formulation of dynamic traffic assignment problem. One of the nice properties of this formulation is that we can always numerically solve a problem before analyzing it. In this sense, the J-system formulation is not only internally related to the BMW formulation (Beckmann et al., 1956), but more general than the latter.

In the definition of J-system of the dynamic TAP, we assume that FIFO violation on each path should be under control, otherwise the computation error in travel times in (6) would adversely affect solutions of the dynamic TAP. In this sense, the commodity-based kinematic wave model of network traffic developed in (Jin, 2003; Jin and Zhang, 2004) is a good candidate, since the error in travel times is to the order of simulation time step, $\Delta t/M$, and its solutions converge to FIFO ones as shown in (Jin, 2003; Jin and Zhang, 2004; Jin and Jayakrishnan, 2005). This well-discussed model is also computationally efficient and mathematically rigorous.

In the future, we will study the existence and uniqueness of dynamic UE in a general network with this formulation and investigate more efficient computational methods for searching UE in large networks. All these studies will be based on better understanding of network traffic dynamics, for which the commodity-based kinematic wave model could be a good tool since it is mathematically more tractable than any other network traffic flow models.

This paper completes our initial discussions of the dynamic system of the traffic assignment problem. We expect these studies to have fundamental and profound impacts on studying transportation systems and other social systems with user equilibria.

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\mathcal{R}	set of origin nodes; $\mathcal{R} \subset \mathcal{N}$
\mathcal{S}	set of destination nodes; $\mathcal{S} \subset \mathcal{N}$
\mathcal{K}_{rs}	set of paths connecting O-D pair $r - s$; $r \in \mathcal{R}, s \in \mathcal{S}$
τ	independent decision variable
$\Delta\tau$	decision step
t	independent time variable
$[0, T_0]$	assignment interval
$[0, T]$	traffic simulation interval
N	the number of assignment time steps
M	the number of traffic simulation steps during each assignment time step
Δt	assignment time step, $\Delta t = T_0/N$
$\Delta t/M$	traffic simulation time step
t_n	assignment time instants, $t_n = n\Delta t$ for $n = 0, \dots, N$
t_i	traffic simulation time instants, $t_i = i\Delta t/M$ for $i = 0, \dots, MNT/T_0$
$f_k^{rs}(r, t)$	arrival flow at origin r at t taking path k connecting O-D pair $r - s$; $\mathbf{f}^{rs}(r, t) = (\dots, f_k^{rs}(r, t), \dots)$; $\mathbf{f}(r, t) = (\dots, \mathbf{f}^{rs}(r, t), \dots)$
$g_k^{rs}(r, t)$	arrival flow-rate at origin r at t taking path k connecting O-D pair $r - s$; $\mathbf{g}^{rs}(r, t) = (\dots, g_k^{rs}(r, t), \dots)$; $\mathbf{g}(r, t) = (\dots, \mathbf{g}^{rs}(r, t), \dots)$
$f_k^{rs}(s, t)$	arrival flow at destination s at t taking path k connecting O-D pair $r - s$;
$g_k^{rs}(s, t)$	arrival flow-rate at destination s at t taking path k connecting O-D pair $r - s$;
$p_{rs}(r, t)$	arrival flow at origin r between origin-pair $r - s$;
$q_{rs}(r, t)$	arrival flow-rate at origin r between origin-pair $r - s$;
$p_{rs}(s, t)$	arrival flow at destination s between origin-pair $r - s$;
$q_{rs}(s, t)$	arrival flow-rate at destination s between origin-pair $r - s$;
$c_k^{rs}(r, t)$	travel time on path k connecting O-D pair $r - s$ for a vehicle arriving at origin r at t ;
$v_{rs}(r, t)$	travel time between O-D pair $r - s$ for a vehicle arriving at origin r at t ;
$\mathcal{F}^{rs}(t)$	the set of $\mathbf{f}^{rs}(t)$ satisfying (1)
$\mathcal{F}(t)$	the set of $\mathbf{f}(t)$; $\mathcal{F}(t) = \prod_{rs} \mathcal{F}^{rs}(t)$
K_{rs}	the number of initially non-empty paths connection O-D pair $r - s$
$J_k^{rs}(r, t)$	FIFO violation of flow from destination r at t on path k connecting O-D pair $r - s$; $\mathbf{J}^{rs}(r, t) = (\dots, J_k^{rs}(r, t), \dots)$; $\mathbf{J}(r, t) = (\dots, \mathbf{v}^{rs}(r, t), \dots)$; $\mathbf{J} = (\dots, \mathbf{J}(r, t_n), \dots)$
$\ \mathbf{J}\ _2$	2-norm of $\mathbf{J}(\mathbf{f})$, defined in (8)

Table 1: Network notation system

TAP	the traffic assignment problem incorporating traffic dynamics
CKW	commodity-based kinematic wave model of network traffic flow (Jin, 2003)
O-D	origin-destination
FIFO	First-In-First-Out
J-functions	FIFO violation functions
J-system	an autonomous dynamic system of path flows in decision variable τ
JE	(dynamic) J-equilibria, steady states of J-system
UE	(dynamic) user equilibria

Table 2: Some abbreviations

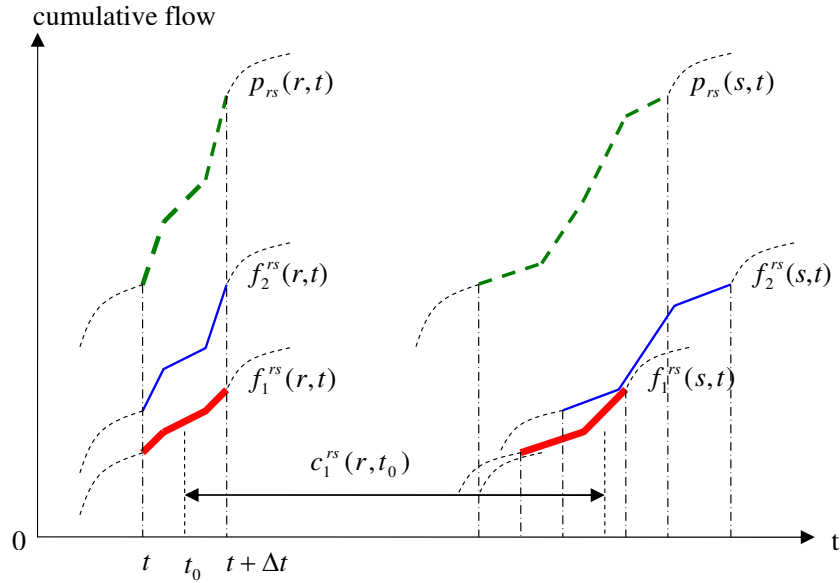


Figure 1: First-In-First-Out violation in dynamic traffic assignment problem

Initialize $T_0, T, \tau, N, M, \Delta t, \Delta \tau$
Initial guesses of $f_k^{rs}(r, t_n) \forall k, r, s, n$
for $\tau_l = \Delta \tau, 2\Delta \tau, 3\Delta \tau \dots$
Use traffic flow model to obtain $f_k^{rs}(s, t_i) \forall k, r, s$ and $i = 1, \dots, MNT/T_0$
Compute travel time $c_k^{rs}(r, t_n)$ and $v_{rs}(r, t_n)$
Compute J-functions $J_k^{rs}(r, t_n) \forall k, r, s$
Use finite different equation to solve $\Delta f_k^{rs}(r, t_n)$ and find $f_k^{rs}(r, t_n) (\forall n = 1, \dots, N)$
Adjust $f_k^{rs}(r, t_n)$ to satisfy (1)

Table 3: Finding JE

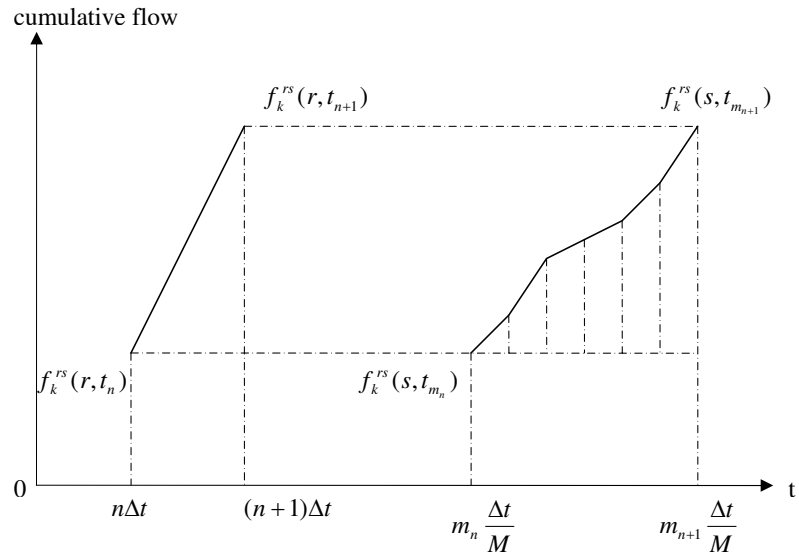


Figure 2: Computation of travel time

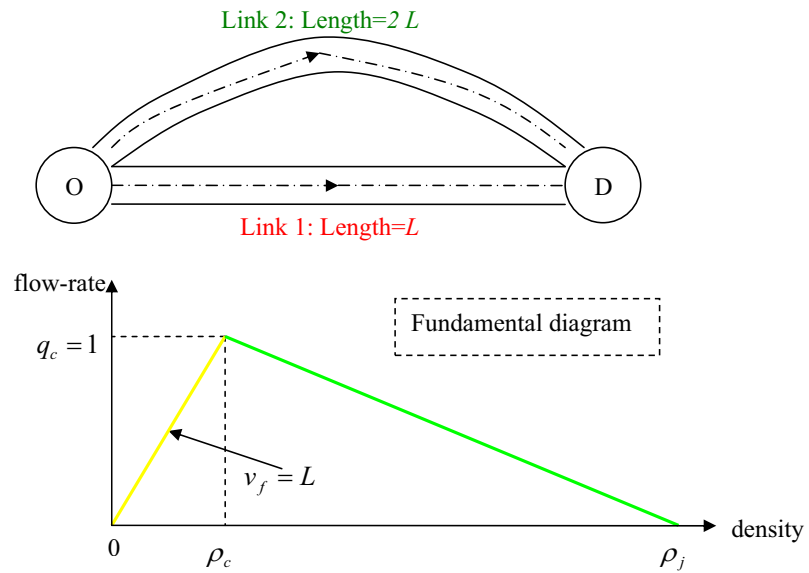


Figure 3: A network for dynamic traffic assignment

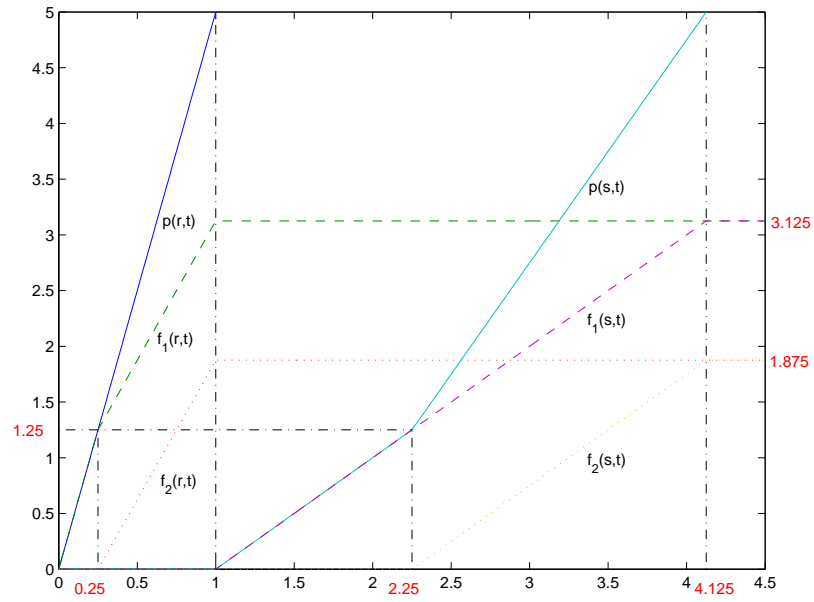


Figure 4: Dynamic UE solutions

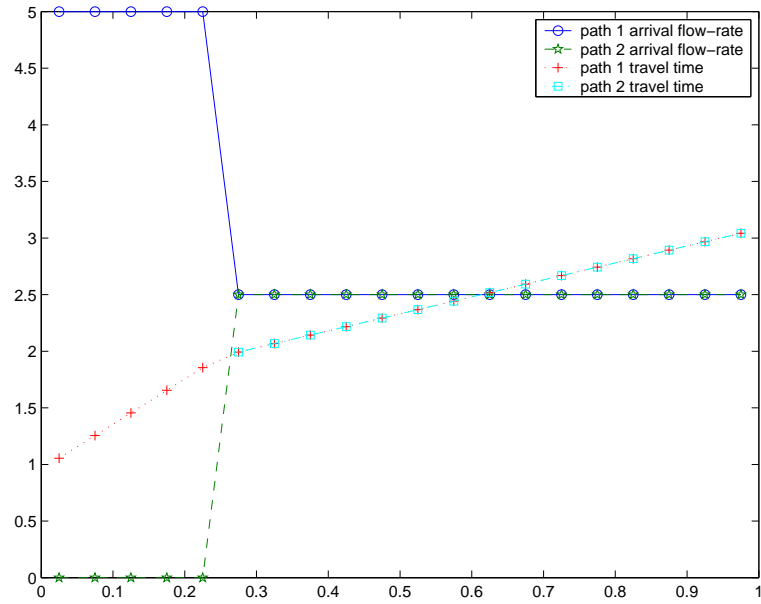


Figure 5: UE arrival flow-rates at the origin and travel times

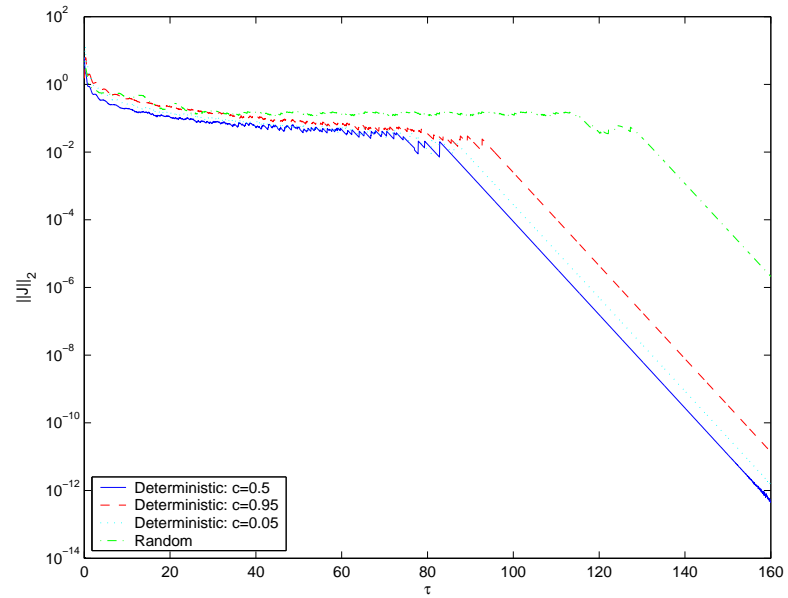


Figure 6: Convergence of solutions to a dynamic traffic assignment problem with different initial time series with $\Delta\tau = 0.05$