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**The Dynamic System of the  
Traffic Assignment Problem:  
Part I Theory**

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## Part I. Theory

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### Abstract

In this paper, we first define so-called J-functions of First-In-First-Out violation, from which we define J-equilibria and a corresponding dynamic model of changing path flows, called J-system. Then we demonstrate that for fixed demand the objective function in the BMW's mathematical programming formulation (Beckmann, McGuire, and Winsten, 1956) can be interpreted as the total energy of J-system. This leads to a new definition of user equilibria as the only globally stable steady states of J-system. In this sense, J-system is *the* dynamic system of the traffic assignment problem. We further show the existence of J-system as *the* dynamic system of the traffic assignment problem with variable demand. From this study, we can see that the new formulation and the BMW formulation are internally related, and the former can have as wide applications as the latter. Then in a following paper, we develop computational methods intrinsic to this simple and intuitive formulation for solving traffic assignment problems. In yet another following paper, we propose J-system formulation of dynamic traffic assignment problem.

**Keywords:** Static traffic assignment problems, Fixed demand, Variable demand, J-functions of First-In-First-Out violation, J-equilibria, J-system, User equilibria, Steady states

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# 1 Introduction

The traffic assignment problem (TAP) has been successfully formulated as mathematical programming problem, variational inequality problem, nonlinear complementarity problem, and fixed point problem (Patriksson, 1994). Smith (1984) proposed yet another formulation, in which user-equilibria (UE) are stable steady states of a dynamic system. However, due to the complexity in constructing the dynamic system, this dynamic system formulation has not been widely used in analyzing and solving TAP.

In this paper, we propose a new dynamic system formulation for static TAP based on the definition of J-functions of First-In-First-Out (FIFO) violation. This dynamic system is referred to as J-system, with which we can have a physical interpretation of the objective function in the mathematical programming formulation proposed by (Beckmann, McGuire, and Winsten, 1956). We will demonstrate that this unique dynamic system can be used to define UE for both fixed demand case and variable demand case.

In this study we adopt the network notation system given in **Table 1**, which is similar to that in (Sheffi, 1984), but with several new definitions. We also have the following basic relationships:

$$q_{rs} = \sum_k f_k^{rs}, \quad \forall r, s \quad (1a)$$

$$f_k^{rs} \geq 0, \quad \forall k, r, s; \quad (1b)$$

and

$$x_a = \sum_{r,s} \sum_k f_k^{rs} \delta_{a,k}^{rs}, \quad \forall a \quad (2a)$$

$$c_k^{rs} = \sum_a t_a \delta_{a,k}^{rs}, \quad \forall k, r, s. \quad (2b)$$

In addition, we use abbreviations in **Table 2**.

The rest of the paper is organized as follows. In Section 2, we define J-functions of FIFO violation, J-equilibria, and J-system. In Section 3, we demonstrate that J-system is *the* dynamic system of the traffic assignment problem. In Section 4, we develop J-system for TAP with variable demand.

## 2 A dynamic system of TAP with fixed demand

### 2.1 FIFO violation functions

As an example, we consider TAP for a network with one O-D pair,  $r-s$ , and two alternative paths. In one assignment,  $(f_1^{rs}, f_2^{rs})$ , one obtains two different path travel times,  $(c_1^{rs}, c_2^{rs})$ . The cumulative arrival curves at the origin and destination (Moskowitz and Newman, 1963; Newell, 1993) are shown in Figure 1, where all vehicles of one path are assumed to arrive at the same time instant. From the figure we clearly see FIFO violation among two path flows. If we assume  $v_{rs}$  as the average travel time, we can obtain measurements of FIFO violation for two path flows,  $(J_1^{rs}, J_2^{rs})$ , which correspond to the areas of the two regions shown in Figure 1, but subject to opposite signs. Note that here the unit of FIFO violation is vehicle $\times$ time. We can also define the average FIFO violation, whose unit is in time, as the total FIFO violation divided by the total O-D demand. Such definitions of FIFO violation are inspired by (Jin and Jayakrishnan, 2005), where measurements of FIFO violation in time and location for vehicles on the same path were developed for a kinematic wave model of network vehicular traffic.

For a general road network, we can define the FIFO violation function, or simply J-function, for path  $k$  connecting O-D pair  $r-s$  by  $(\forall k, r, s)$

$$J_k^{rs} = q_{rs} f_k^{rs} (c_k^{rs} - v_{rs}), \quad (3a)$$

whose unit is vehicle<sup>2</sup> $\times$ time.<sup>1</sup> Since the average O-D travel time is

$$v_{rs} = \frac{\sum_j f_j^{rs} c_j^{rs}}{q_{rs}}, \quad (3b)$$

J-function can be re-written as

$$J_k^{rs} = f_k^{rs} (q_{rs} c_k^{rs} - \sum_j f_j^{rs} c_j^{rs}) = f_k^{rs} \sum_j f_j^{rs} (c_k^{rs} - c_j^{rs}). \quad (4)$$

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<sup>1</sup>Here we multiply FIFO violation defined in Figure 1 by  $q_{rs}$  for the purpose of clarity, and our discussions apply when it is multiplied by  $(q_{rs})^n$  for any integer  $n$ .

## 2.2 J-equilibria

**Definition 2.1 (J-equilibria (JE))** *Roots of equations  $\mathbf{J}(\mathbf{f}) = \mathbf{0}$  are called J-equilibria (JE). That is,  $\mathbf{f} \in \mathcal{F}$  is JE iff for  $\mathbf{J}(\mathbf{f}) = \mathbf{0}$ .*

**Theorem 2.2 (Equivalent definitions of JE)** *The following six definitions of JE are equivalent for  $\mathbf{f} \in \mathcal{F}$ :*

1.  $\mathbf{f}$  satisfies

$$J_k^{rs} = f_k^{rs} \sum_j (c_k^{rs} - c_j^{rs}) f_j^{rs} = 0, \quad \forall k, r, s \quad (5)$$

2.  $\mathbf{f}$  satisfies

$$J_k^{rs} = q_{rs}^2 \xi_k^{rs} \sum_j (c_k^{rs} - c_j^{rs}) \xi_j^{rs} = 0, \quad \forall k, r, s, \quad (6)$$

where  $\xi_k^{rs} = f_k^{rs}/q_{rs}$  is the proportion of flow on path  $k$  among flow on O-D pair  $r-s$ .

3.  $\mathbf{f}$  satisfies

$$f_k^{rs} (c_k^{rs} - v_{rs}) = 0, \quad \forall k, r, s \quad (7)$$

4.  $\mathbf{f}$  satisfies

$$c_k^{rs} = c_j^{rs} \text{ when } f_k^{rs} f_j^{rs} \neq 0 \quad \forall k, j, r, s \quad (8)$$

5.  $\mathbf{f}$  satisfies

$$(c_k^{rs} - c_j^{rs})^n f_k^{rs} f_j^{rs} = 0, \quad \forall k, j, r, s \quad (9)$$

for any  $n = 1, 2, \dots$

6.  $\mathbf{f}$  satisfies

$$c_1^{rs} \leq \dots \leq c_j^{rs} \leq c_{j+1}^{rs} = \dots = c_{j+l}^{rs} \leq c_{j+l+1}^{rs} \leq \dots, \quad (10a)$$

$$f_k^{rs} > 0, \quad \forall k = j + 1, \dots, j + l, \quad (10b)$$

$$f_k^{rs} = 0, \quad \forall k = 1, \dots, j, j + l + 1, \dots \quad (10c)$$

That is, in JE, taken paths share the same travel time, but un-taken paths may have longer or shorter travel time.

**Remarks.** The proof of this theorem is straightforward and omitted. Note that, (5) and (6) are groups of nonlinear equations, which in general have multiple solutions. That is, we usually have multiple JE of J-system. For example, any vertex in the polygon defined by (1) is a JE.

### 2.3 A dynamic system

Assuming that all path flows depend on a decision variable  $\tau$ , whose unit is  $(\text{vehicle} \times \text{time})^{-1}$ , we propose a dynamic model for changing path flows as follows:

$$-\dot{f}_k^{rs} = J_k^{rs} = f_k^{rs} \sum_j (c_k^{rs} - c_j^{rs}) f_j^{rs} = q_{rs} f_k^{rs} (c_k^{rs} - v_{rs}), \quad \forall k, r, s \quad (11a)$$

where  $\dot{f}_k^{rs}$  is the derivative of  $f_k^{rs}$  with respect to  $\tau$ . That is, if the travel time of path  $k$  is larger than the average travel time on O-D pair  $r - s$ , we decrease its flow. If discretizing  $\dot{f}_k^{rs}$ , we can have

$$\dot{f}_k^{rs} = \frac{f_k^{rs}(\tau + \Delta\tau) - f_k^{rs}(\tau)}{\Delta\tau}. \quad (11b)$$

That is, in the discretized system, path flows are changed with decision step  $\Delta\tau$ . In a dynamic model of shifting flows from one path to another developed in (Smith, 1979), route-choice behavior of a driver is considered as day-to-day. We can also interpret (11b) similarly, by assuming the decision variable as a date, and  $\Delta\tau$  equal to one day. However, we prefer to consider  $\tau$  as an abstract variable, which can be interpreted in other ways, such as the number of iterations in a numerical algorithm for solving the dynamic system. That is,  $\tau$  is related to time but independent of time.

From (11a), we can have an autonomous system of ordinary differential equations (Strogatz, 1994)

$$-\dot{\mathbf{f}} = \mathbf{J}(\mathbf{f}), \quad (12)$$

which is referred to as J-system. Thus, steady states of J-system are JE. One property of (12) is that, if  $f_k^{rs}(0) = 0$ , then  $f_k^{rs}(\tau) = 0$  for any  $\tau > 0$ ; that is, if a path is initially not used, it is not used at any decision step.

### 3 User equilibria as stable steady states of J-system

#### 3.1 Traditional definitions of user equilibria

**Definition 3.1 (User equilibria (UE))** We call  $\mathbf{f}$  as a user-equilibrium (UE) in the sense of (Wardrop, 1952) iff for  $\mathbf{f} \in \mathcal{F}$

$$f_k^{rs}(c_k^{rs} - v_{rs}) = 0, \quad \forall k, r, s \quad (13a)$$

$$c_k^{rs} \geq v_{rs}, \quad \forall k, r, s. \quad (13b)$$

As we know, UE are the solutions of the following mathematic programming problem (Beckmann et al., 1956)

$$\min z(\mathbf{x}) = \sum_a \int_0^{x_a} t_a(\omega) d\omega, \quad (14)$$

whose objective function is called the BMW objective function.

For traditional definitions of UE, we have the following remarks:

1. Extensive studies of formulations and methods on UE can be found in literature (Patriksson, 1994).
2. For separable link cost functions, there exists a unique UE solution in  $\mathbf{x}$ . However, it has been shown that UE solutions in  $\mathbf{f}$  might not be unique (Sheffi, 1984). For other types of link cost functions, such as asymmetric and capacitated ones, existence and uniqueness are conditional.

3. The BMW objective function  $z(\mathbf{x})$  for fixed demand case is strictly convex in  $\mathbf{x}$  for increasing link performance functions (Dafermos and Sparrow, 1969); and the objective function with asymmetric link performance functions is strictly convex for increasing link performance function and a positive Jacobian matrix (Dafermos, 1971; Smith, 1979).

In our study, we hereafter assume that  $z(\mathbf{x})$  is convex, but not necessarily strictly convex. That is,  $z(\mathbf{x})$  still attains its minimum at the UE solutions, but UE solutions may not be unique in  $\mathbf{x}$ .

**Theorem 3.2** *Assuming that  $\mathbf{x}^*$  is a UE, then  $z(\mathbf{f}) \equiv z(\mathbf{x}(\mathbf{f})) \geq z(\mathbf{x}^*)$  and  $\mathbf{f}$  is a UE  $\iff z(\mathbf{f}) = z(\mathbf{x}^*)$ .*

**Remarks.** This is a restatement of the mathematical programming formulation of UE.

### 3.2 UE as globally stable JE

It is straightforward that, if  $\mathbf{f}$  is a UE, then it is a JE, but the reverse statement is generally not true. That is, UE are steady states of J-system, but some steady states may not be UE.

Long since the introduction of the mathematical formulation by (Beckmann et al., 1956), the meaning of the BMW objective function has been mystical. As stated in (Sheffi, 1984, Section 3.1): “This function does not have any intuitive economic or behavioral interpretation. It should be viewed strictly as a mathematical construct that is utilized to solve equilibrium problems.” Here we provide a physical interpretation of the BMW objective function.

**Theorem 3.3 (Lyapunov function of J-system)** *The function*

$$V(\mathbf{f}) = z(\mathbf{f}) - z(\mathbf{x}^*) \tag{15}$$

*is a Lyapunov function (LaSalle, 1960; Strogatz, 1994) of J-system at UE. That is*

1.  $V(\mathbf{f}) > 0, \forall \mathbf{f} \in \mathcal{F}$  and  $\mathbf{f}$  is not a UE;
2.  $V(\mathbf{f}) = 0$  iff  $\mathbf{f} \in \mathcal{E}$ , where  $\mathcal{E}$  is the set of UE;
3.  $\text{grad } V(\mathbf{f}) \cdot \mathbf{J}(\mathbf{f}) \leq 0$  if  $\mathbf{f}$  is not a UE.

*Proof.* This is given in Appendix A.

Therefore, the BMW objective function can be considered as the total energy of J-system, which attains its minimum at UE.

**Theorem 3.4 (A stability theorem of J-system)** *J-system, (12), has the following stability properties:*

1. *J-system is locally, asymptotically stable around  $\mathcal{E}$  in the sense that solutions of (12) converge to a UE for a non-UE initial state that is close enough to  $\mathcal{E}$ .*
2. *A JE that is not UE is not stable in  $\mathcal{F}$ , but may be stable in a subset of  $\mathcal{F}$  that satisfies (1).*

*Proof.* This is given in Appendix B.

From Theorem 3.4, we can see that UE are the only steady states of J-system that are globally stable in the whole set  $\mathcal{F}$ .

**Definition 3.5 (A new definition of UE)**  *$\mathbf{f}$  is a UE iff it is a steady state of J-system, (12), and it is globally stable in  $\mathcal{F}$ . That is, a UE is a globally stable JE of J-system.*

**Remarks.** The following remarks are in order.

1. The mathematical programming formulation is to minimize the energy function of J-system.
2. Since the BMW objective function is the total energy of J-system and UE are its only stable steady states, J-system can be considered as *the* dynamic system of TAP.
3. The new definition of UE can be used to find UE solutions, as shown in a following paper.

## 4 Discussions

In this paper, based on the definition of J-functions of FIFO violation, we defined J-equilibria and a dynamic model of changing path flows, called J-system. Then we demonstrated that the BMW objective function in the mathematical programming formulation can be interpreted as the total energy of J-system and that user equilibria are its only globally stable steady states. In this sense, J-system can be considered as *the* dynamic system of the traffic assignment problem.

Since system-optimal assignments are equivalent to user-equilibrium assignments with modified link performance functions, we can also have similar J-system formulation of system-optimal assignments. The new formulation of TAP can be extended for formulating and solving many other traffic assignment problems. For example, we can extend it for variable demand case, whose mathematical programming formulation is given by (Beckmann et al., 1956; Sheffi, 1984)

$$\min z(\mathbf{x}, \mathbf{q}) = \sum_a \int_0^{x_a} t_a(\omega) d\omega - \sum_{rs} \int_0^{q_{rs}} u_{rs}(\omega) d\omega, \quad (16a)$$

subject to (1) and

$$q_{rs} \geq 0, \quad \forall r, s, \quad (16b)$$

where  $u_{rs}(q_{rs})$  is a given, decreasing travel time function associated with O-D pair  $r-s$  and is the inverse of the demand function. It has been shown that the BMW objective function of (16a) is still convex under certain conditions, and there exists a unique user equilibrium in  $\mathbf{x}^*$  and  $\mathbf{q}^*$ . We define the following J-system

$$\dot{f}_k^{rs} = -q_{rs} f_k^{rs} (c_k^{rs} - u_{rs}), \quad \forall k, r, s \quad (17a)$$

$$\dot{q}_{rs} = -q_{rs} \left( \sum_k f_k^{rs} c_k^{rs} - q_{rs} u_{rs} \right), \quad \forall r, s \quad (17b)$$

Or if we introduce the average O-D travel time  $v_{rs}$  as in (3b), we can have an equivalent J-system

$$\dot{f}_k^{rs} = -f_k^{rs} \sum_j (c_k^{rs} - c_j^{rs}) f_j^{rs} - q_{rs} f_k^{rs} (v_{rs} - u_{rs}), \quad \forall k, r, s \quad (18a)$$

$$\dot{q}_{rs} = -q_{rs}^2(v_{rs} - u_{rs}), \quad \forall r, s \quad (18b)$$

Then we say that J-system, (17) or (18), is *the* dynamic system of the traffic assignment problem with variable demand in the sense that: (1) the BMW objective function in (16a) is the total energy function of J-system (17); and (2) user equilibria of the problem are the only globally stable steady states of J-system. In Appendix C, we prove the first statement but omit the proof of the second, which is the same as that in Appendix B.

Note that the idea of dynamically changing path flows was also used in (Smith, 1984), where the dynamic model was not so simple and straightforward as (17) or (18). Actually, one can construct many dynamic models with UE as their steady states, but J-system is unique, since it is stable at UE and its energy function is the well-known BMW objective function. With such internal connections with the BMW's mathematic programming formulation, the J-system formulation can have as wide applications as the BMW formulation, including theoretical investigations of stability, existence, and sensitivity of UE, numerical computations of user equilibria and system optimal solutions, and even dynamic traffic assignment. In (Jin, 2005b), we develop computational methods intrinsic to this simple and intuitive formulation for solving traffic assignment problems. In (Jin, 2005a), we propose J-system formulation of dynamic traffic assignment problem.

## Appendix A: Proof of Theorem 3.3

From the definitions of the Lyapunov function  $V(\mathbf{f})$  in (15) and the BMW objective function  $z(\mathbf{f})$  in (14), we can easily see the first two statements are correct. We now prove the third statement in detail.

We first compute the gradient of  $V(\mathbf{f})$  with respect to  $\mathbf{f}$ , whose  $(rs, k)$ th element is ( $\forall k, r, s$ )

$$\frac{\partial V(\mathbf{f})}{\partial f_k^{rs}} = \frac{\partial z(\mathbf{x})}{\partial f_k^{rs}} = \sum_a t_a(x_a) \delta_{a,k}^{rs} = c_k^{rs}.$$

Therefore,

$$\text{grad } V(\mathbf{f}) \cdot \mathbf{J}(\mathbf{f}) = \sum_{rs} \sum_k \frac{\partial V}{\partial f_k^{rs}} f_k^{rs} = - \sum_{rs} \sum_k c_k^{rs} f_k^{rs} \sum_j (c_k^{rs} - c_j^{rs}) f_j^{rs},$$

which leads to

$$\text{grad } V(\mathbf{f}) \cdot \mathbf{J}(\mathbf{f}) = - \sum_{rs} \sum_k \sum_{j>k} (c_k^{rs} - c_j^{rs})^2 f_k^{rs} f_j^{rs} \leq 0, \quad (19)$$

since all path flows are non-negative. From the definition of JE in (9), we can see that  $\text{grad } V(\mathbf{f}) \cdot \mathbf{J}(\mathbf{f}) = 0$  iff  $\mathbf{f}$  is a JE. For non-JE state,  $\text{grad } V(\mathbf{f}) \cdot \mathbf{J}(\mathbf{f}) < 0$ . Therefore,  $V(\mathbf{f})$  is a Lyapunov function of J-system. ■

## Appendix B: Proof of Theorem 3.4

**Lemma 1** *The set of JE is closed. That is, given a sequence  $\mathbf{f}_i \rightarrow \mathbf{f}^*$ , if all  $\mathbf{f}_i$  are JE, then  $\mathbf{f}^*$  is also a JE.*

*Proof.* Since  $(\forall k, r, s)$

$$c_k^{rs}(\mathbf{f}) = \sum_a \delta_{a,k}^{rs} t_a \left( \sum_{mn} \sum_l \delta_{a,l}^{mn} f_l^{mn} \right)$$

is continuous in  $\mathbf{f}$ , J-functions,  $\mathbf{J}(\mathbf{f})$ , are continuous in  $\mathbf{f}$ . Then, with  $\mathbf{f}_i \rightarrow \mathbf{f}^*$ ,  $\mathbf{J}(\mathbf{f}_i) = 0$  implies  $\mathbf{J}(\mathbf{f}^*) = 0$ . That is,  $\mathbf{f}^*$  is also a JE, and the set of JE is closed. However, note that the set of JE is generally not connected, since we may have many pieces of JE. ■

**Lemma 2** *The set of UE,  $\mathcal{E}$ , is closed and connected.*

*Proof.* Given a sequence of  $\mathbf{f}_i \in \mathcal{E}$  and  $\mathbf{f}_i \rightarrow \mathbf{f}^*$ , then from continuity of the BMW objective function, we can see that  $z(\mathbf{f}_i) \rightarrow z(\mathbf{f}^*)$ . Therefore,  $z(\mathbf{f}^*)$  is also minimum, and  $\mathbf{f}^* \in \mathcal{E}$ . Thus  $\mathcal{E}$  is closed. Since  $z(\mathbf{x})$  is a convex function and  $\mathbf{x}$  is in a convex set, the set of UE,  $\mathcal{E}$ , is connected in  $\mathbf{x}$ . Further, since  $x_a = \sum_{rs} \sum_k \delta_{a,k}^{rs} f_k^{rs}$  is a continuous mapping from  $\mathcal{F}$  to  $\mathcal{A}$ , we have that  $\mathcal{E}$  is also connected in  $\mathbf{f}$ . ■

*Proof of Theorem 3.4.* From the lemmas above, we can find a region around UE, where all non-UE states are not JE. Then from the proof in Appendix A, we can see that  $\text{grad } V(\mathbf{f}) \cdot \mathbf{J}(\mathbf{f}) < 0$  for all non-UE states. Therefore, according to the Lyapunov's stability theorem in (Smith, 1984), asymptotic solutions of (12) converge to UE solutions for non-UE initial states. That is, the set of UE is asymptotically stable. Note that we can have multiple UE and cannot use the traditional Lyapunov's stability test. Thus, the first statement is proved.

If  $\mathbf{f}$  is a JE but not UE, there exists an empty but shorter path  $k$  for some O-D pair  $r-s$ . Then, for a taken path  $j$ ,  $c_k^{rs} < c_j^{rs}$ ,  $f_k^{rs} = 0$ , and  $f_j^{rs} > 0$ . We introduce a perturbation to this JE by shifting  $\epsilon_0$  ( $f_j^{rs} > \epsilon_0 > 0$ ) from path  $j$  to path  $k$ . We denote the resultant state by  $\tilde{\mathbf{f}}$  and the corresponding vector of path travel times by  $\tilde{\mathbf{c}}$ . Since path travel times are continuous in  $\mathbf{f}$ , we can have an  $\epsilon_0$  small enough such that path  $k$  is still shorter than other taken paths; i.e.,  $\tilde{c}_k^{rs} < \tilde{c}_j^{rs}$  and  $\tilde{c}_k^{rs} < \tilde{c}_l^{rs}$  for any  $f_l^{rs} > 0$ . Then at  $\tilde{f}_k^{rs} = \epsilon_0$ , the local changing direction in  $\tilde{f}_k^{rs}$  is

$$\dot{\tilde{f}}_k^{rs} = -\tilde{f}_k^{rs} \sum_l (\tilde{c}_k^{rs} - \tilde{c}_l^{rs}) \tilde{f}_l^{rs} > 0.$$

That is, the perturbed state  $\tilde{\mathbf{f}}$  will drift away from the original JE, and the original JE is not stable. Thus, the second statement is proved. However, in a subset of  $\mathcal{F}$ , in which flows on shorter paths are always zero, such a state may be stable. Actually,  $\mathcal{F}$  is completely stable in the sense of (LaSalle, 1960) that, if an initial state is in  $\mathcal{F}$ , all other states are. ■

## Appendix C: Proof of that the BMW objective function in (16a) is the total energy of J-system (17)

*Proof.* First, as shown in Appendix A, ( $\forall k, r, s$ )

$$\frac{\partial z(\mathbf{x}, \mathbf{q})}{\partial f_k^{rs}} = c_k^{rs}.$$

Then, ( $\forall r, s$ )

$$\frac{\partial z(\mathbf{x}, \mathbf{q})}{\partial q_{rs}} = -u_{rs}.$$

Thus,

$$\begin{aligned} \text{grad } z(\mathbf{x}, \mathbf{q}) \cdot \mathbf{J}(\mathbf{q}, \mathbf{f}) &= \sum_{rs} \left( \sum_k \frac{\partial z(\mathbf{x}, \mathbf{q})}{\partial f_k^{rs}} \dot{f}_k^{rs} + \frac{\partial z(\mathbf{x}, \mathbf{q})}{\partial q_{rs}} \dot{q}_{rs} \right) \\ &= - \sum_{rs} \left( \sum_k c_k^{rs} q_{rs} f_k^{rs} (c_k^{rs} - u_{rs}) - u_{rs} q_{rs} \left( \sum_k f_k^{rs} c_k^{rs} - q_{rs} u_{rs} \right) \right) \\ &= - \sum_{rs} q_{rs} \sum_k f_k^{rs} (c_k^{rs})^2 - 2 f_k^{rs} c_k^{rs} u_{rs} + f_k^{rs} (u_{rs})^2 \\ &= - \sum_{rs} q_{rs} \sum_k f_k^{rs} (c_k^{rs} - u_{rs})^2, \end{aligned}$$

which is always non-positive and is zero iff  $(\mathbf{q}, \mathbf{f})$  is a JE of J-system (17). Therefore, the BMW objective function in (16a) is the total energy of J-system (17).  $\blacksquare$

## A note on the development of the idea and acknowledgement

I came up with the idea of applying FIFO violation in studying the traffic assignment problem after developing measurements of FIFO violation in (Jin and Jayakrishnan, 2005). I got some preliminary results in this direction on December 14, 2004. However, the original approach was very complicated, and not much effort was put on it. On January 28, 2005, Prof. David Boyce of Northwestern University gave a talk on traffic assignment problems in a seminar here at the Institute of Transportation Studies of UC Irvine (UCI-ITS), and during discussion time I mentioned about the idea of making use of FIFO violation in numerically solving the traffic assignment problem. After the presentation, I had some numerical experiments with Matlab. Still, this topic did not look so interesting to me at that time. My view was changed on February 4, 2005, when I met Hyunmyung Kim on the aisle outside of my office, and he said he was interested in my FIFO idea when he heard about it in the seminar. Then we had a prolonged discussion, and I found that the idea could have some interesting applications. After that, I spent a great deal of time on reviewing literature and probing different aspects of the topic of FIFO violation. Finally I

reached J-system when trying to find solution methods for JE on February 22, 2005 and conjectured that the BMW objective function be the Lyapunov function of J-system later on that day. On February 23, I came up with the proof in Appendix A, from which the proof in Appendix B becomes straightforward.

I greatly appreciate Mr. Hyunmyung Kim of UCI-ITS for getting me back interested in the topic of FIFO violation; without him, it would probably take longer for me to have these interesting results. I gratefully appreciate my supervisor, Prof. Will Recker, director of UCI-ITS, for providing me a friendly research environment. I also appreciate Prof. Jayakrishnan of UCI-ITS for some general discussions on the traffic assignment problem. The views and results contained herein are the author's alone.

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$\mathcal{N}$	node (index) set
$\mathcal{A}$	arc (index) set
$\mathcal{R}$	set of origin nodes; $\mathcal{R} \subset \mathcal{N}$
$\mathcal{S}$	set of destination nodes; $\mathcal{S} \subset \mathcal{N}$
$\mathcal{K}_{rs}$	set of paths connecting O-D pair $r - s$ ; $r \in \mathcal{R}, s \in \mathcal{S}$
$\tau$	decision variable, independent of time
$\Delta\tau$	decision step
$x_a$	flow on arc $a$ ; $\mathbf{x} = (\dots, x_a, \dots)$
$t_a$	travel time on arc $a$ ; $\mathbf{t} = (\dots, t_a, \dots)$
$f_k^{rs}$	flow on path $k$ connecting O-D pair $r - s$ ; $\mathbf{f}^{rs} = (\dots, f_k^{rs}, \dots)$ ; $\mathbf{f} = (\dots, \mathbf{f}^{rs}, \dots)$
$c_k^{rs}$	travel time on path $k$ connecting O-D pair $r - s$ ; $\mathbf{c}^{rs} = (\dots, c_k^{rs}, \dots)$ ; $\mathbf{c} = (\dots, \mathbf{c}^{rs}, \dots)$
$\xi_k^{rs}$	the proportion of flow on path $k$ among flow on O-D pair $r - s$
$q_{rs}$	traffic demand between origin-pair $rs$ ; $\mathbf{q} = (\dots, q_{rs}, \dots)$
$v_{rs}$	average travel time for O-D pair $r - s$
$u_{rs}$	travel time function for variable demand for O-D pair $r - s$
$\delta_{a,k}^{rs}$	indicator variable: $\delta_{a,k}^{rs} = \begin{cases} 1 & \text{if link } a \text{ is on path } k \text{ between O-D pair } r - s \\ 0 & \text{otherwise;} \end{cases}$
	$\Delta^{rs} = (\dots, \delta_{a,k}^{rs}, \dots)$ ; $\Delta = (\dots, \Delta^{rs}, \dots)$
$\mathcal{F}^{rs}$	the set of $\mathbf{f}^{rs}$ satisfying (1)
$\mathcal{F}$	the set of $\mathbf{f}$ ; $\mathcal{F} = \prod_{rs} \mathcal{F}^{rs}$
$J_k^{rs}$	FIFO violation of flow on path $k$ connecting O-D pair $r - s$ ;
	$\mathbf{J}^{rs} = (\dots, J_k^{rs}, \dots)$ ; $\mathbf{J} = (\dots, \mathbf{J}^{rs}, \dots)$
$\mathcal{E}$	the set of user equilibria

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Table 1: Network notation system

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TAP	the traffic assignment problem
O-D	origin-destination
BMW	(Beckmann, McGuire, and Winsten, 1956)
FIFO	First-In-First-Out
J-functions	FIFO violation functions
J-system	an autonomous dynamic system of path flows in decision variable $\tau$
JE	J-equilibria, steady states of J-system
UE	User equilibria

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Table 2: Some abbreviations

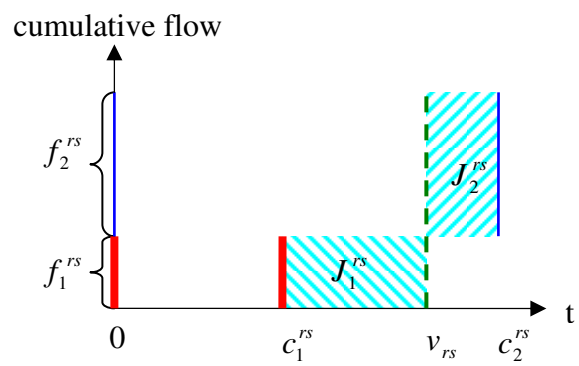


Figure 1: An example of First-In-First-Out violation