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Abstract

Utility theory views traffic flow as a process of human behavior in maximization of individual utility. This paper proposes a utility function for drivers and makes two observations and an assumption. Some properties of the utility function are derived. The research shows that there is no optimal state of traffic flow for drivers to maintain. This provides a starting point for a future research on traffic flow.

Keywords: Traffic flow, utility function, fundamental relationship

INTRODUCTION

Traffic flow theory is fundamental to traffic control. Fundamental to traffic flow theory is the basic relationship between flow rate, speed and concentration, the so called *fundamental relationship*. Zhang (1) presented three ways to deduce these fundamental relationships of traffic flow: (a) from statistical modeling (b) from car-following and (c) from fluid analogies. Typical work in the first category is described in Prigogine and Herman (2), Prigogine (3), and Herman, Lam and Prigogine (4). A number of well-known functional forms for speed and concentration are derived from steady-state car-following theories. They include the Greenshields' linear model (5), Greenberg's logarithmic model (6), Underwood's exponential model (7). Falling into the third class of models is the famous kinematic theory of traffic flow developed by Lighthill and Whitham (8) and Richard (9) (known as LWR theory). Zhang (10) represents a more recent development along this direction. All the theories that fall into the three categories try to dictate a unique fundamental relationship, which does not apply under certain conditions, for example, under congested traffic. Because of this, during the past few years, there has been considerable interest in research on traffic flow under congestion. Among the most recent examples are work by Kerner (11), and Zhou and Hall (12).

The study of human behavior could be rewarding in the exploration of fundamental relationship since traffic flow is the result of human behavior. Further, through the study of human behavior, we can explain some traffic phenomena such as platoon dispersion that traditional theories on the fundamental relationship cannot. Theories on driver's behavior are important in another sense. They are very useful in

micro-simulation models that have developed in the past decade (for example, Jayakrishnan, Mahmassani and Herman, 13).

In this paper, a qualitative study based on utility theory is conducted. As a starting point for further research, it addresses the following question: is there a unique optimal state of traffic flow for drivers to maintain? Here, by state of traffic flow, we mean a flow of a certain density and speed.

The organization of this paper is as follows. First a driver's utility function is analyzed. Then two basic observations are made, followed by a proposition that shows there is no unique optimal flow state. We end with a conclusion.

INDIVIDUAL UTILITY FUNCTION

Driving is more of a human behavior that maximizes either an individual driver's or drivers' collective utility than it is the movement of a mineral particle or continuum fluid. The utility includes primarily the minimum time that the driver is trying to achieve in reaching the destination, and the comfort of driving, which is related to en-route safety and relaxation.

Then total utility is broken up into two parts,

1. Utility of time U_t
2. Utility of comfort U_c

U_t is related to speed. It is a function of travel time, hence of speed. U_c is the comfort at driving. It reflects the relaxation, and is a function of both speed and gap.

We need to express utility in term of observable physical variables. The basic elements are gap and speed. *Gap* is the distance between two cars next to each other. *Speed* is the distance that the vehicle covers in unit time. There are also unobservable variables such as anticipated speed and perception of safety gap. These are considered exogenous here.

Figure 1 shows relationship between observable variables and utility components.

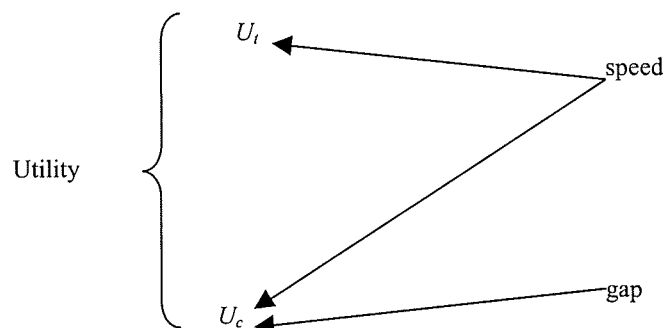


Figure 1. Components of a driver's utility and their relationship to speed and gap

In the following section, we make some basic observations about the properties of the utility function.

OBSERVATION I

U_t is assumed qualitatively to be in the form shown in figure 2.

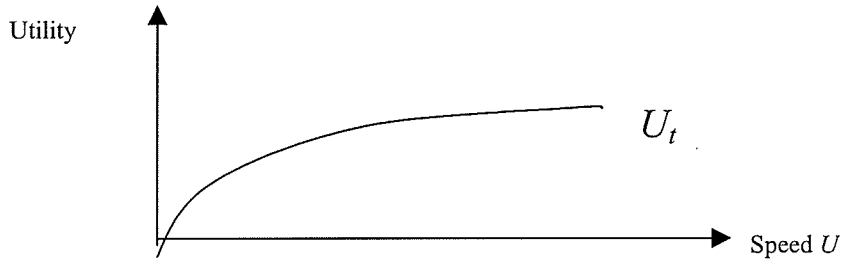


Figure 2. Observed relationship between U_t and u .

Mathematically, the utility function U_t is concave with respect to speed u . i.e.

$\partial^2 U_t / \partial u^2 \leq 0$, but also there exists the following condition: $\partial U_t / \partial u \geq 0$. This simply

says the faster, the better. But the marginal utility gained through raising speed decreases.

This can be shown mathematically if we assume U_t is proportional to travel time. Then

$$U_t = k/u + \text{const.} \quad \text{where } k < 0$$

By taking the second derivative with respect to u , it can be easily seen that U_t is concave in u .

Secondly, we assume that U_t does not change with gap g .

To summarize, Observation I can be expressed in the following equations.

$$\partial U_t / \partial g = 0 \quad (1)$$

$$\partial^2 U_t / \partial u^2 < 0, \quad (2)$$

$$\partial U_t / \partial u > 0, \quad (3)$$

Equations (1) through (3) lead to the following results.

Conclusion I. Utility U_i is concave in the two-dimensional space of speed and gap.

Proof

This is obtainable through Hessian matrix by further considering the fact that $\partial^2 U_i / \partial u \partial g = 0$.

The observation above is quite natural and understandable.

Corollary 1. If a driver is *only* concerned about the utility of time, his speed will tend to infinity.

A proof for this can be obtained from equation (3). Increase of speed can always bring about an increase of utility.

OBSERVATION II

This observation is concerned with the utility from the comfort in driving. They are expressed in the following equations.

$$\partial U_c / \partial u < 0, \quad (4)$$

$$\partial U_c / \partial g > 0 \quad (5)$$

$$\partial^2 U_c / \partial u^2 \leq 0 \quad (6)$$

$$\partial^2 U_c / \partial g^2 \leq 0 \quad (7)$$

Equation (4) says that utility U_c decreases with speed. Equation (5) states that U_c increases with gap. Equation (6) indicates that the decrease of utility becomes bigger and bigger with speed. Equation (7) says that the increase rate of utility U_c decreases with gap.

Conclusion II

If the driver *only* cares about the utility U_c , the optimal state must be that the driver does not move at all and the gap tends to infinity.

This simply says that the slower, the better; and the larger the gap is, the better. Conclusion II can be drawn from the observations on first order partial derivative to both u and g , as in Equation (4) and (5).

An assumption

If we assume that function U_c is continuous and twice differentiable, and based on observation, we further assume that U_c is concave, then we must have this equation.

$$(\partial^2 U_c / \partial u \partial g)^2 \leq \partial^2 U_c / \partial g^2 * \partial^2 U_c / \partial u^2 \quad \forall u \geq 0, g \geq 0 \quad (8)$$

The above condition is a sufficient and necessary condition for U_c to be concave based on the observations made earlier. Its concavity says that for any three points 1, 2 and 3 on a line in domain of speed u and gap g , the last point always falls below the projection through the first two points in terms of utility U_c . The concavity can be explained in this way. Along any direction to improve individual utility, the increase in U_c is smaller and smaller. In addition, this precludes the existence of saddle points. A saddle point just implies that at some point except global optimum, it is impossible to improve utility U_c by adjusting either speed or gap. We simply assume saddle points do not exist and that at ANY point, drivers can improve their utility by adjusting speed or gap if they are not at the optimum under the condition that speed and gap are both positive. The assumption here stipulates that a stationary point, if exists, must be global optimal point.

BASIC RESULTS

In this section, we derive some basic results based on the observations above.

Proposition I

If it is assumed that the utility consists of two parts U_c and U_l only, based on observation I, II and assumption (8), the utility function is concave.

Proof

Let $U = U_t + U_c$. Based on observation I and II, both components of the utility are concave in the two dimensional space of gap and speed. So, their sum is also concave. Since U_t is strictly concave, henceforth, we can say that the sum is also strictly concave.

Proposition I holds even in the case that there are more linear constraints over speed and gap. It is a common practice that the speed is subject to a maximum limitation. That is, $u \leq u_L$.

Of interest is the fact that traditional theories just describe the relationship between speed and density as a (curved) line while utility theory treats the relationship as a surface. Speed and gap define a two dimensional space where drivers maximize their individual utility.

Corollary 2

There is no stationary state of traffic flow for drivers to maintain given a utility function defined in Proposition I.

Proof

Given that the utility function is concave, the optimal point must be found at stationary points. But from U_t and U_c , we know that the first partial derivative with respect to gap g is always greater than zero. It means drivers will always be better off by enlarging their gaps.

The reason, not often observed in reality, that gaps tend toward infinity is that there are rare cases where the driver has enough space and time to evolve fully and that behavior is not homogeneous among drivers. An example showing that drivers tend to enlarge their gaps is platoon dispersion.

On the other hand, given a constant gap, there must be an optimal speed to maximize the utility. A conjecture is that these optimal speeds at given gaps are about those dictated by traditional theories on fundamental relationships to correspond to the given densities.

CONCLUSION

In this paper, a driver's utility function is studied. Two basic observations are made. Based on the observations and a reasonable assumption, we can show that there is no stationary optimal flow state for drivers to maintain if they have full freedom. To conclude, we believe that utility theory is important for both micro-simulation and for the study of fundamental relations of speed and density. This paper serves as a starting point for the authors research on this topic. Continuing research includes the study of specific forms of the utility functions and possibly some new principles in traffic control for collective utility maximization.

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