Discrete Allocation Models with Asymptotically Optimal Properties for Network Revenue Management

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ABSTRACT

Discrete allocation models are fundamental to development of a series of network revenue management policies whose applications span airline, hotel and rental car services. In this paper, we present two classes of models, stochastic and deterministic, with asymptotically optimal properties. The models we propose add modeling flexibility to allow us to approximate special characteristics of the problem such as nesting effects. They highlight a potential for improvement to the models currently in use. In this paper, we also examine the asymptotic optimality of two other stochastic models studied by other researchers. These, along with other recent models, form a library of asymptotically optimal discrete allocation models.

Key Words: Network Revenue Management, Asymptotic Optimality, Discrete Allocation
INTRODUCTION

Revenue management applied to perishable assets over a finite period of time has wide practical application in industries such as airline, hotel and automobile rental. It specifies the conditions under which customer demands are satisfied to maximize the total revenue before the product perishes. If there are multiple products and a consumer consumes one or more products at a time, the problem can be defined as a network yield management problem. In the area of hotel and rental car services, a product is just a provision of one-day service of a single unit of resource (room or car). And in airline yield management, a product is a seat on a single flight. The network yield management problem has very significant practical applications.

There have been several classes of methods for network yield management. The first is the discrete allocation method. This method basically controls the yield management process by allocating the limited number of products to different demand classes. Demand is satisfied at a certain price level when an allocation is available. This method corresponds to the make-to-stock policy examined in Gallego and Van Ryzin (1997). It ignores the dynamics between early and late demand and uses the aggregate demand information for the entire sales period. Research of this type includes Curry (1990), Gallego and Van Ryzin (1997), Ciancimino et al. (1999) and Cooper (2002). Models of this type have great computational applicability to large-scale optimization problems. They are capable of dealing with the complicated network effects of problems such as the airline revenue management problem over a large service network. However the discrete allocation method is often subject to the problem where an allocation computed could be a fraction of a product. Curry (1990) provides a virtue nesting method to overcome this problem. It maintains its mathematical maneuverability and optimality by assuming conditions such as low fare demands arrive before high fare demands. However, that model also has good performance when the assumption of demand arrival order is relaxed. The second method is the bid price control method. Examples in this area include Talluri and Van Ryzin (1998) who show an asymptotically optimal bid price control policy and Bertsimas and Popescu (2003) who study an adaptive and non-additive bid price method. In addition, Talluri and Van Ryzin (1999) study a randomized linear programming method to estimate the gradient of the perfect information network revenue. Chen et al (1998) represents a third class of methods which study the network revenue management with Markovian theory and regression splines. Due to the complexity of the dynamic network problem, almost all methods to solve it borrow from discrete allocation models to approximate the optimal resource consumption. As a result, discrete allocation models are fundamental to the development of various policies.

In contrast to the network yield management problem is the single leg yield management problem. Following Littlewood (1972), examples of the research of this type include Belobaba (1987, 1989), Gallego and Van Ryzin (1994), Feng and Xiao (2000), Liang (1999), Brunelli et al (1990), and Wollmer (1992). We refer to this type of research as dynamic yield management. Methods in Markovian theory, birth-and-death processes, intensity control theory and dynamic programming are typical tools in the research of this type. However, the dynamic yield management methods generally lack the capability to
solve the network yield management problem. They are therefore applied to a leg-based yield management system that ignores the network effect in resource consumption. These used to be a popular practice in the airlines industry and are still used in some airlines. But a general trend now is to transit to an origin-destination based yield management system to account for the network effects.

Even though discrete allocation models are fundamental to network yield management, there has not been much work done in this area. In a network yield management setting, Gallego and Van Ryzin (1997) showed that a simple deterministic discrete allocation model assuming a Markovian process is asymptotically optimal. Cooper (2002) explains the underlying mechanism for that result. This model, though easy to implement, ignores the difference in price levels and demand intensities in limiting the allocation to be no more than the expected demand in each class. We extend it into a class of asymptotically optimal models that are capable of modeling the difference in price and demand intensities. In a similar spirit, we examine an alternative stochastic model similar to the one used in Ciamicimino et al (1999), our study shows that a model used in Ciamicimino et al (1999) for railway yield management is also asymptotically optimal. However, the stochastic model we examine does not have the capability to capture the nesting effect. We develop a class of asymptotically optimal models as an extension with the flexibility to overcome this drawback. In addition, based on the models we discuss, we show the asymptotic property of early static network models as in Curry (1990). The condition for the asymptotic property in the models we propose here differs from those in Gallego and van Ryzin (1997) and Cooper (2002) in that we require a weaker condition. Some of our models do not require a Markovian process as in Gallego and Ryzin (1997). Neither do they require scalable demand and scalable resources as in Cooper (2002). These models add to the collection of asymptotically optimal discrete allocation models which have both practical and theoretical value.

Though not the focus of our research, the models we examine also show computational feasibility because of their separable concave objective functions. The models we examine generally have a computational complexity that falls between the models such as Curry (1990) and the simpler models presented in Gallego and Van Ryzin (1994) and Cooper (2002).

**AN ASYMPTOTICALLY OPTIMAL MODEL**

Without loss of generality, we position this problem in the context of airline yield management. The problem is defined as follows. There are $m$ flights on a service network, combinations of which can be used to create $n$ itineraries. Assume there is a finite time horizon during which sales can be made. We define the end of the time horizon as Time 0, and $t$ time units prior to Time 0 as Time $t$. Over the period $[t, 0]$, consumer requests for tickets on the itineraries arrive according to a stochastic process. Each arriving customer is assumed to belong to one of the $n$ itineraries. A customer who requests itinerary $j$ is defined as customer (demand) $j$. The objective is to decide the
conditions under which to satisfy a demand on each itinerary in order to maximize the total revenue.

The method to allocate the seats into different fare buckets to maximize the total revenue is called the discrete allocation method. The discrete allocation method is typically adopted to deal with the airline yield management problem over a large service network. A demand is satisfied when a seat allocation to itinerary is available. The complexity involved in this method is the consideration of the nesting effect. Since market differentiation is exercised, multiple itineraries at different fare levels covering the same set of flights could be defined. A class nest is a set of such itineraries. The nesting effect refers to the consumption of a seat allocated to low fare demand by a high fare passenger within the same class nest when the seats allocated to high fare passengers have been used up. Seats allocated to high fare passengers are never allowed to be used for low fare passengers.

We define the notation first.

\[ A = m \times n \] matrix. \( A = \{ a_{ij} \} \), where \( a_{ij} = 1 \) represents itinerary over flight \( i \).

\( C \) the number of seats of the \( m \) flights in a vector form.

\( p_i \) the fare for bucket \( i \).

\( \pi \) a revenue management policy. \( \pi_j(t) = 1 \) represents a policy \( \pi \) which allows the sale of one ticket on itinerary \( j \) at time \( t \).

\( N^{\pi_j}(t) \) the number of accepted requests for itinerary \( j \) during the period \([t,0]\) under policy \( \pi \).

\( D_{j}(t) \) the number of customer requests for itinerary \( j \) during the period \([t,0]\).

\( \nu(t) \) the optimal revenue from Time \( t \) till Time \( 0 \).

\( \Pi \) the set of all allowable policies.

In addition, all vectors here are column vectors. Therefore, the optimal yield management policy is one defined as follows.

\[
\sup_{\pi} \int \sum p_{ij} dN^{\pi_j}(x) = \sup_{\pi} \left( \sum p_{ij} d\int N^{\pi_j}(t) \right)
\]

Subject to

\( AN^{\pi_j}(t) \leq C \) \hspace{1cm} (0)

Here \( N^{\pi_j}(t) = (N^{\pi_j}(t)) \) is a column vector. And the discrete seat allocation policy is one in the following form.

\[
\max \sum p_{ij} \nu_j(x) = \max \sum p_{ij} \nu_j(x)
\]

Subject to \( \nu x \leq C \) \hspace{1cm} (1)
Where $R_j(c)$ is the expected revenue of itinerary $j$ from a seat allocation $x$. Expansion of this term is often challenging because of the nesting effect.

Gallego and van Ryzin (1997) demonstrate the asymptotic property of a simple linear program as follows when a Markovian demand increases to infinity with time.

$$\max_x \left\{ p \cdot x : Ax \leq C, 0 \leq x \leq \mu \right\}$$

(II)

Cooper shows the asymptotic behavior of the discrete allocation policy from program (II). Cooper (2002) relies on the following conditions. (a) The sales period is scalable; and, (b) The number products available increase proportionally with the number of sales periods.

Here we start with discuss an alternative model under a weaker condition. This model has a concave non-linear objective function in hopes to achieve a balance between model accuracy and computational simplicity.

We discuss the following mathematical program for discrete seat allocation.

$$\max \ p_j \ \int \frac{xf_j(u)}{2} \ du + \int \frac{x}{x_j} f_j(u) \ du$$

(III)

Subject to

$$Ax \leq C$$

We denote by $f_j(u)$ the distribution of aggregate demand $j$ during the period $[t, c]$ and by $x_j$ seat allocation to demand $j$. Further, we denote by $NP(t)$ the objective value of program (III) given a feasible seat allocation $x$, by $NP(t)$ the optimal objective value in program (III), and by $R^{se}(t)$, the expected revenue with seat allocation $x$ when nesting effect is considered with demand arrival order relaxed. For simplicity, we denote $R^{se}(t)$ by $R^{se}(t)$ when the optimal allocation from program (III) is adopted.

When the demand follows a discrete distribution, the objective function in program (II) can be replaced with the following.

$$p_j \left[ \sum_{n \in j} x_j F_j(n) + \sum_{n \in j} x_j F_j(n) \right]$$

(1)
Where \( P_j(n) \) is the probability of having \( n \) demand in bucket \( j \). Our analysis focuses on the case with continuously distributed demand though we believe the analytical results here apply to the case whose demand follows a discrete distribution.

We clarify an assumption first.

**Assumption 1**

The expected demand tends to infinity when time also tends to infinity. That is,

For any limited value \( U \geq 0 \), there holds

\[
\lim_{\nu \to \infty} \frac{\int_0^U u f_\nu^{(j)}(u) \, du}{\nu} = 0, \forall j
\]

Many situations justify this assumption. If it is assumed that demand for this product exists at any time prior to the end of the sales season, the total expected demand amounts to infinity when the time horizon is also infinite and therefore the assumption is satisfied.

One can verify this in a parallel case with discrete demand by assuming that the demand \( j \) at any time \( r > 0 \) is a Poisson process with intensity \( \lambda(r) > 0 \) such that

\[
\lim_{\nu \to \infty} \int_0^\infty \lambda(t) dt = +\infty.
\]

In that case, \( \lim_{\nu \to \infty} \sum_{n=\nu}^\infty \nu P(n) = 0 \) for any limited value \( K \) where \( P(n) \) is the probability of having \( n \) demands. The conditions assumed in Gallego and Van Ryzin (1997) and Cooper (2002) also satisfy this assumption in the discrete case.

**Lemma 1**

Given a fixed allocation \( x_j \) for itinerary \( j \), the following holds under Assumption 1.

\[
\lim_{\nu \to \infty} \sum_{n=\nu}^\infty P_j \left( \int_0^u f_\nu^{(j)}(u) \, du + \int_{x_j}^u x_j f_\nu^{(j)}(u) \, du \right) = p_j x_j
\]

(2)

**Proof**

Since \( f_\nu^{(j)}(u) \geq 0, \forall u \), together with Assumption 1, we conclude that for any fixed value \( u \geq 0 \), there holds \( \lim_{\nu \to \infty} f_\nu^{(j)}(u) = 0 \).

After slight algebraic operation, we have the following,

\[
\int_0^u f_\nu^{(j)}(u) \, du + \int_{x_j}^u x_j f_\nu^{(j)}(u) \, du = x_j + \int_0^{x_j} (u - x_j) f_\nu^{(j)}(u) \, du
\]
Since $x_i$ is a limited number (no larger than the resource defined by $C_i$) and the number of itineraries is limited as well, we must have the following result.

$$\lim_{t \to \infty} \sum_{i=1}^{t} \left( (u - x_i) f_i(u) \right) du = 0$$

The result (2) therefore holds.

(End of proof)

**Lemma 2**

The following holds.

$$NF(i) \leq R^m(i) \leq \mu(i) \leq p - x^m \quad \forall t$$

Where $x^m$ is the optimal solution to the following mathematical program,

$$\max \ p \cdot x \quad \text{(IV)}$$

Subject to

$$Ax \leq C$$

**Proof**

The first inequality holds because the expected objective value of program (II) does not consider the nesting effect. The second inequality is true by definition. We then show the third inequality.

Under the optimal policy, denote by $N^m$ the number of demand accepted. Since $AN^m \leq C$ holds, we must have $AE(N^m) \leq C$. Therefore, $E(N^m)$ is a solution to the linear program (IV). Since $z^m$ is the optimal solution to program (IV), we have $p \cdot E(N^m) \leq p \cdot z^m$. So the third inequality holds.

(End of proof)

The following proposition shows that the seat allocation policy resulting from program (III) is asymptotically optimal.

**Proposition 1**

$$\lim_{t \to \infty} R^m(t) = \mu(t).$$

**Proof**
When $t \to \infty$, program (III) reduces to program (IV) based on Lemma 1. As a result, we have $\lim_{t \to \infty} N_P(t) = p \cdot x^\infty$. Combined with Lemma 2, the conclusion holds.

(End of proof)

The condition required to obtain the asymptotically optimal result is weaker than in Gallego and Van Ryzin (1997). Compared with the simple linear program (II), program (III) looks complex in its objective function. However, a nice property is that program (III) has a separable concave objective function. One can easily verify this by deriving the Hessian matrix. Therefore, a class of general concave programming methods applies to it. A similar model is used in Cincimino et al. (1999) for railway revenue management with one more constraint set, $l \leq x$ to ensure a minimum number of tickets are allocated to each itinerary. In light of our proof here, any model with an extra fixed number of side constraints included in program (III) is asymptotically optimal such as the one in Cincimino et al. (1999).

Two Classes of Asymptotically Optimal Models

We construct two classes of models with asymptotic property in the following section. One class includes stochastic models and the other deterministic ones.

**Stochastic Models**

From the proof for Proposition 1 and Lemma 2, we are able to conclude that program (IV) is asymptotically optimal. Therefore, program (IV) and program (III) delineate a space where a class of concave programs have asymptotic property. We present this result in Proposition 2.

**Proposition 2**

The following program is asymptotically optimal for any $\alpha_j$, where $0 \leq \alpha_j \leq 1$ under Assumption 1.

$$\max \sum_j \left( p_j x_j - \alpha_j \int_0^{\infty} p_j (u - x_j) f_j(u) du \right)$$

**Subject to**

$$Ax \leq C$$

**Proof**

The objective value for program (V) is equivalent to the following.

$$\sum_j \left( \alpha_j \int_0^{\infty} p_j \alpha f_j(u) du + \int_0^{\infty} p_j x_j f_j(u) du + (1 - \alpha_j) p_j x_j \right)$$
whose value is at between those for program (III) and (IV). The conclusion follows.

(Please note that it is not required to have $\alpha_i = \alpha_j, i \neq j$. We call this class asymptotically optimal stochastic model in this paper. To our observation, if an objective function approximates with high accuracy the expected revenue from a certain seat allocation policy, the model would be expected to generate a good seat allocation. These models seem capable of adjusting approximation to the expected revenue from the seats allocated to each fare bucket in its objective function. For some fare bucket $j$, program (III) may underestimate the expected revenue while program (IV) overestimates it for a given seat allocation $x_j$. For others, the opposite might hold. Program (V) leaves much flexibility to fine tune the model. This flexibility is important in cases where large volume of demand is handled on a daily basis such that a small difference in modeling leads to a significant revenue improvement. Airline revenue management is such a typical example. The fact of ignoring the nesting effect by models in the form of program (III) has been a serious concern. Model (V) provides the necessary freedom to approximate the expected revenue when the nesting effect is taken into account. To its advantage, the concavity of the objective function of program (V) ensures its mathematical maneuverability. A significant work may be necessary to calibrate the model’s parameters. This could be done through computer simulation. The parameters $\alpha$ could be a function of the fare structure or relative demand across the buckets depending on particular applications. Therefore, we do not provide a general calibration method.

Remarks

There are alternative models with asymptotic properties such as the following program:

$$\max_{x_j} \sum_{j} \left[ 2p_j x_j - \left( \int_{0}^{x_j} p_j s_j(u)du + \int_{x_j}^{1} p_j s_j(u)du \right) \right] : Ax \leq C$$

Further, we can say that any program lying between program (III) and the above one is asymptotically optimal. However, its disadvantage is that the objective function is not concave.

Deterministic Models

In contrast to the stochastic models that could be complicated to implement and could be computationally expensive, simple linear programming models clearly have computational advantages in applications. We further specify a class of deterministic models with asymptotic property in the following proposition.

Lemma 3

Program (IV) is asymptotically optimal.
This result can be obtained from the proof for Proposition 1.

**Proposition 3**

The following models are asymptotically optimal under the condition that the demand process is Poisson and the expected demand for each fare bucket tends to infinity with time.

\[
\begin{align*}
\text{Max} & \quad p \cdot x \\
\text{Subject to} & \quad Ax \leq C \\
& \quad x \leq \mu_i
\end{align*}
\]  

(VI)

Where \( \mu_i = (\mu_i') \) and \( \mu_i = \beta_i \mu_i \) with \( \beta_i \geq 1, \forall i \). \( \beta \) is an exogenous coefficient.

**Proof**

In Gallego and Van Ryzin (1997), program (II) is shown to be asymptotically optimal under the same condition. It is easily seen that the optimal objective value for program (VI) is always at between those for program (II) and (IV). In light of the asymptotic property of programs (II) and (IV), the conclusion holds.

(End of proof)

As can be seen that program (II) does not have the capability of considering the fare difference between buckets. By introducing \( \beta \) into the formulation, some errors associated with fixing the \( \mu \) value might be reduced. In the following hypothetical example, we show that in some cases both the stochastic and deterministic models proposed have advantage over program (II) analyzed in Gallego and van Ryzin (1997) and in Cooper (2005).

**Example 1** Suppose there are only two seats available at Time 1 on the flight. There are two fare buckets offered: $10 and $2 each. The demand arrives according to a Poisson process. The demand intensities for both buckets are \( \lambda_1 = \lambda_2 = 1 \).

Here, we have \( A = (1, 1), x = (x_1, x_2), C = 2, \mu_i = \mu_2 = 1 \) and \( p_1 = 10, p_2 = 2 \). Obviously program (II) leads to a seat allocated to each bucket while the optimal allocation should be to allocate both seats to the bucket at $10. The underlying reason for the sub-optimal allocation is that program (II) is not capable of balancing the demand intensity and price. On the contrary, if we adopt program (VI) by setting \( \beta_1 = 2, \beta_2 = 1 \), program (VI) leads to the optimal solution. It demonstrates that program (VI) has the capability to pay special consideration to the difference in demand and fare. Further, one can verify that by using
program (III) in the discrete case (the objective function is replaced with Equation (1)), an optimal solution is also obtainable.

While this example shows an advantage of the models proposed, we do not claim that program (II) developed by Gallego and van Ryzin (1997), is inferior. There may be many other cases in which program (VI) just takes the form of program (II). However, we do have it as our conclusion here that it is important to explore alternative models with asymptotic properties in order to identify those that perform well in various circumstances and that it is important to make a thorough comparison of the alternative models.

Please note that side constraints sometimes need to be added to be equitable. It is noteworthy that program (IV), (V) and (VI) still possess the asymptotic property with side constraints such as those considered as in Ciancimino et al (1999).

Based on the model we propose, we will further show the inherent connection between the early static models and dynamic optimal ones. We take Curry (1990) as an example.

The Asymptotic Property of Curry's Model

There could be many stochastic models with asymptotic properties in addition to the two classes specified above. Curry (1990) is such an example. Curry (1990) proposes a discrete seat allocation model for network revenue management. The model is described as follows.

\[
\max \sum R^k(x^k) \quad (VII)
\]

subject to \( Ax^k \leq C \)

Where \( x^k \) is the seat allocation to \( k \)-class nest, and \( R^k \) is the expected revenue of \( k \)-class nest.

Curry (1990) considers the nesting effect. Under the condition that low fare passengers arrive before high fare passengers, Curry (1990) shows that \( R^k \) is concave. Therefore, the program is a concave program and methods of its kind can be adopted and an optimal policy is derived. Curry (1990) reports good numerical results when the resulting seat allocation is applied in a simulation environment with the demand arrival order relaxed. We thereby analytically examine the optimality of Curry's (1990) model when the demand arrival order condition is relaxed.

Let denote by \( CR \) the objective value in program (VII) from seat allocation \( x \), and \( CR \) the optimal objective value from Curry's model (VII). Again, if we let \( u(t) \) represent the revenue under the optimal policy, we have \( CR_x \leq u(t) \).

We have the following result.
Lemma 3

\[ NP(t) \leq CR(t) \leq \alpha(t) \leq p \cdot x^{\alpha(t)} \]

The first inequality results from the fact that program (VII) partially considers the additional revenue when low fare seat allocation is available and high fare demands have used up high fare allocations.

Proposition 2

Program (VII) is asymptotically optimal. That is \( \lim_{t \to \infty} CR(x) = p \cdot x^{\alpha(t)} \)

Proof

Since \( \lim_{t \to \infty} N^*(t) = p \cdot x^{\alpha(t)} \), Lemma 3 leads to the above result.

(End of proof)

From the proof above, we may see that models that capture partial nesting effect can be proved asymptotically optimal in the same way.

CONCLUSION

Network yield management has significant applications in practice. Airline revenue management on large service networks is a typical example. Discrete seat allocation models are fundamental to the development of network yield management policies. However, there has not been much work on discrete seat allocation models. To date, only three models with asymptotic properties have been presented. A problem with those models is that none of them render much flexibility in consideration of the differences of demand intensity and fare structure as well as the nesting effect. In this paper, we present two classes of asymptotically optimal models. We build the flexibility into the model to capture different characteristics in the fare structure and demand intensity. This is very important in cases where nesting effect exists and where there is significant difference in demand intensity and price levels between demand segments that consume the same resources. We further show that Curry’s (1990) model is asymptotically optimal when the demand arrival order requirement is relaxed. Generally, the models we propose require a weak condition in terms of demand process for the asymptotic property to hold.

Our models add onto a family of asymptotically optimal models first examined by Gallego and van Ryzin (1997) and later by Cooper (2002). They enhance the flexibility to model different circumstances. The models we propose also bring challenges about how to calibrate the extra parameters. We believe these parameters are application specific and therefore leave this work to future users when the necessary information is available for calibration.
REFERENCES