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ABSTRACT

In the commercial airline industry, aircraft assignments are subject to changes because of irregular operations and the practice of re-assigning planes to alternative routes due to unexpectedly high (or low) demand known as demand driven dispatching. Therefore there is a need for yield management that explicitly incorporates the possibility of future capacity changes. Though there is a rich literature on the topic of yield management in the context of stochastic demand, no published research to date deals with the case in which capacity is also stochastic. In this paper, we consider a simple case in which at some point prior to departure, assignments are subject to changes. We decompose the problem into two stages, present an optimal solution and compare it with solutions to the related problem in which future capacity changes are not anticipated. We examine these relative to their optimal control policies and their revenue potential. An upper bound estimate for the optimal revenue is presented as well. We further build a yield management heuristic for the problem with two swappable aircraft embedded in the fleet assignment where aircraft swapping can be determined any time prior to departure. In addition to commercial airline operations, this problem has wide practical applications in rental car and hotel management.

Key words: Yield Management; Dynamic Programming; Demand Driven Dispatching; Irregular Airline Operations

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INTRODUCTION

Consider a booking process for a flight that starts at time $T$ before departure time ($t = 0$) in which there are $N$ distinct fare classes with fares $F_1, F_2, \ldots, F_N$ and in which we assume that $F_i > F_j$ for $i < j$. The corresponding competing booking processes of these classes are assumed to be independent inhomogeneous Poisson processes (no multiple seat bookings) with intensity functions $\mu_1(t), \ldots, \mu_N(t)$. These are known as incremental booking curves. It is assumed that a request for a seat that has been denied will not be made later in another fare class. A yield management policy is one that determines when to open or shut down fare classes so as to maximize the expected revenue for each flight. At time $T_0$ ($T_0 > 0$), the aircraft might be reassigned, effectively changing the capacity of the flight. We assume that for any flight there are exactly two potential capacities, $c_1$ and $c_2$, with $c_1 < c_2$, and, that these capacities have corresponding assignment probabilities $p$ and $(1-p)$. A yield management method that anticipates this potential future capacity change is needed.

In the airline industry, aircraft assignments are subject to changes due to fluctuations in demand. In practice, demand driven dispatching (referred to in the industry as $D^3$) specifically deals with the stochasticity of demand. It tries to make fleet assignments which allow for more opportunities for re-optimization and re-fleeting. We refer to Berge and Hopperstad (1993) for early work on this topic. A drawback of that important early work was that it did not consider the practical constraints related to crew and maintenance scheduling. In addition, Berge and Hopperstad (1993) does not investigate the yield management problem with explicit anticipation of future capacity change. In practice, airline crew and maintenance scheduling is typically done after fleet assignments are fixed. In addition to the scale of the optimization problems that must be solved, other complicated constraints related to legal concerns, labor contracts and other typical practical considerations prohibit an airline from repeatedly exercising a full scale re-optimization from scratch. All these reasons lead to the recent attempts of fleet assignment which limit aircraft swapping opportunities to those involving two cockpit compatible fleets. Examples of this effort see Talluri (1996) who explores a simple algorithm for same-day aircraft swapping and studies the application of his algorithm in the development of airline schedules. Other examples include Paster (1999) who discusses the practice of aircraft swapping as part of demand driven dispatching at Continental Airlines. Other major airlines in the world are making great effort to advance this practice as well. This attempt to make fleet assignments flexible to market demands brings an additional challenge to yield management. To our knowledge, there has been no research to date explicitly dealing with the yield management problem when aircraft swapping is possible.

Because of practical considerations, aircraft assignment changes are typically made about two weeks prior to departure ($T_0 = 14$ days). While the aircraft swapping is a decision, and therefore the capacity of the aircraft can be thought of as a decision variable, for the purposes of pre-swapping yield management, the
capacity can be thought of as a random variable. In general, assignments are fixed after that though mechanical failures or bad weather can lead to last minute changes. To facilitate understanding, we present an example here.

Assume that there are two flights, A and B, whose aircraft are cockpit compatible and swappable. The demands on both A and B are stochastic. Assume that the mean demands for A and B are 100 and 120 respectively and that the total demand is normally distributed. Now suppose we tentatively assign a smaller aircraft to flight A and a larger aircraft to flight B. Traditional yield management assumes a fixed capacity and does not anticipate the opportunity brought about by fleet re-assignment. In fact, if we assume the standard deviation of the demand is 20 for both flights, the probability that flight A will have a higher demand than flight B is approximately 23%. Ignoring the impact from the resulting yield management policy and assuming the aircraft swapping will definitely occur if the demand for A exceeds that of B, there is about 77% probability for market B to have a larger aircraft and 23% probability for market A to have a larger aircraft. Therefore, yield management on flight A shall anticipate 23% probability of capacity upgrade and flight B 23% downgrade. In line with the problem statement, for market B, p equals to 23% while p equals to 77% for market A. With different p values, a different optimal yield management policy applies to the two flights.

For mathematical simplicity, we assume that the future reassignment of aircraft is independent of the future booking process though we note that in practice, re-assignment can be the result of the specific booking policy in use. In this paper, we propose an optimal yield management policy for this problem. In a later part of this paper, we will clarify the practical implication of this assumption.

This problem has significant practical implications. Profit margins in the commercial airline industry are so thin that even a most modest improvement in total revenue could be critical to improving profitability. Related applications include rental car and hotel room management. In both of those applications, yield management must deal with uncertain future resources due to stochasticity of service durations.

RELATED LITERATURE

The last decade has seen the evolution of yield management from early static models towards dynamic ones. Static models such as those presented in Belobaba (1987, 1989), Brumelle et al (1990), Curry (1990) and Wollmer (1992) do not consider the dynamics of the booking process. Later research recognized that booking curve information could be used to capture additional revenue in that it allows a more accurate control of the seat inventory. This understanding led to the development of dynamic programming models. Early dynamic models required the bookings to follow a lower class before a higher class pattern as seen in Brumelle and McGill (1993). Lee and Hersh (1993) relaxed this requirement on demand arrival order and built a discrete time dynamic programming model. Gallego and Van Ryzin (1994), Papastavrou,
Rajagopalan and Kleywegt (1996) and Liang (1999) study dynamic yield management from slightly different angles. Gallego and Van Ryzin (1994) adopt intensity control theory and study the structural monotonicity property. The authors examine the asymptotic optimal fixed price policy as well. Papastavrou, Rajagopalan and Kleywegt (1996) model the problem as a knapsack problem with stochastic demand, which results in a yield management problem with booking curves for different classes taking on the same shape. Liang (1999) specifically focuses on a continuous dynamic yield management and provides an optimal solution for that problem. More recently, Feng and Xiao (2000) study a continuous dynamic yield management problem that assumes only one fare class is available at a time and they give a closed form optimal solution. Following Gallego and Van Ryzin (1994), the authors also use a booking curve for each fare class, which is essentially a nested demand. However, none of the previous studies addresses the case of stochastic future capacity. All of them assume the inventory (capacity) available is deterministic while the demand is uncertain.

The problem we define here is similar to that studied in Liang (1999) in which the yield management problem is formulated as a continuous time, stochastic dynamic programming model and in which an expression for the expected revenue in terms of the stochastic booking process and the control policies is derived. However, our work differs in that we consider the effect of potential aircraft capacity changes after the yield management process has begun. An important issue is what capacity should be assumed during the period prior to permanent aircraft assignment. Yield management based on an explicit a priori capacity assignment cannot be adopted directly to solve this new problem.

Our contributions are three fold. First, we define a more general yield management problem modeled after industry practice, and we extend the basic concepts and results from Liang (1999) to develop a solution for this problem. Our solution takes Liang's as a special case. The proposed solution procedure should shed light on other yield management problems with uncertain future capacity. Next, we examine the impact of this stochastic capacity assignment on the optimal control policy and the total revenue relative to the case where a priori capacity is assumed. We show that optimal yield management associated with a priori capacity could easily be sub-optimal in the context of stochastic assignment. We further provide an estimate (upper bound) on the optimal revenue obtainable when aircraft swapping is allowed.

We decompose the problem into two stages and present solution techniques for these two stages separately. Our primary focus and main contributions concern the first stage that covers the time period prior to the moment when aircraft swapping occurs. For this period, we begin by deriving the salvage function for the dynamic yield management problem. A salvage function evaluates the revenue potential of the capacity remaining at the moment before the capacity assignment is fixed. A discussion of how to determine the initial starting capacity for yield management is also presented. After introducing the optimal solution method for the period prior to aircraft reassignment, we compare the optimal control policy with the case
without anticipation of aircraft reassignment. The revenue improvement resulting from the optimal policy is examined as well. Finally, we present an estimate (upper bound) for the revenue earned under the optimal policy.

**SOLUTION**

We refer to the period after time $T_0$ as period II, and the period before $T_0$ as period I. The period II problem is a stochastic dynamic yield management problem with fixed capacity. That problem has already been solved successfully by Liang (1999). We only need to examine the period I problem in which aircraft capacities are uncertain. We assume that aircraft re-assignment is performed at time $T_0$. We denote the moment just before aircraft reassignment as $T_0^*$, and the moment just after reassignment as $T_0^+$ (Please note that the + and − here refer to remaining time prior to and after the decision instant). To avoid notational confusion, $R_d(\cdot)$ and $R_u(\cdot)$ are used as the revenue functions with and without capacity change respectively.

For period I, we adopt the notation $R_d(T_0^+,c)$ as the salvage function where $c$ represents the capacity available. The salvage function is derived under the optimal policy for period II, which will be explained in more detail later. The period I problem is a dynamic and stochastic yield management problem with uncertain capacity. Because we draw on the method presented earlier by Liang (1999), we first provide his basic ideas here.

Denote a policy by $\Theta_j$. Let $\Theta_j(t, c) = 1$ when class $i$ is open and $\Theta_j(t, c) = 0$ when class $i$ is closed at time $t$ with available capacity $c$. The accepted booking processes are described by the intensities $\mu_{ij}^\theta(t, c) = \mu_j(t)\Theta_j(t, c)$, which are also inhomogeneous Poisson processes. The probability of getting a booking in $dt$ at time $t$ in class $i$ is described as $\mu_{ij}^\theta(t, c)dt$. The probability of getting the first booking in $dt$ at time $t$ in class $i$ is $\exp\left(-\int_t^T \sum_j \mu_{ij}^\theta(\tau)d\tau\right)\mu_{ij}^\theta(t)dt$. First here means first among all bookings, not just first in class $i$. This formula can be used to express optimal expected revenue inductively in terms of the capacity $c$. Once the first booking is accepted, there are $c-1$ seats remaining.

Now we present the solutions for periods II and I.

**The Period II Problem**

The period II problem differs from the one in Liang (1999) in that we extend the domain of $c$ from $[0, c_1] \cup [c_1 - c_2, c_1]$. Let $R_d(t, c)$ be the optimal expected revenue that can be realized from time $t$ to 0
when there are \( c \) seats left at time \( t \). The recursive expression for the expected revenue under policy \( \theta \) is as follows.

\[
R^\theta_H(t, c) = \int_0^t d\tau \exp[\int_0^\tau \sum_{j=1}^N \mu_j(\tau', \tau, c) \sum_{j=1}^N \mu_j(\tau', \tau, c - 1) \mu_j(\tau, c)] \quad \text{when } c > 0 \tag{1a}
\]

and

\[
R^\theta_H(t, c) = Gc \quad \text{when } c \leq 0 \tag{1b}
\]

where \( G \) is the penalty for each seat overbooked.

The derivative with respect to \( t \) can be expressed as follows.

\[
\dot{R}^\theta_H(t, c) = \sum_{j=1}^N \left[ F_j - R^\theta_H(t, c) + R^\theta_H(t, c - 1) \mu_j(t, c) \right] \quad \text{when } c > 0 \tag{2a}
\]

and

\[
\dot{R}^\theta_H(t, c) = 0 \quad \text{when } c \leq 0 \tag{2b}
\]

In the following, we briefly introduce the solution method developed by Liang (1999) for \( R_H(t, c) \) when \( c > 0 \).

The bid price can be expressed as follows. \( \lambda(t, c) = R_H(t, c) - R_H(t, c - 1) \) and possesses the following property: \( d\lambda(t, c)/dt \geq 0 \); for \( c > 1 \), \( \lambda(t, c) \leq \lambda(t, c - 1) \) where \( t \in [0, T_0] \). The optimal control policy \( \Theta(t, c) = 1 \) if \( F_i > \lambda(t, c) \) and is 0 otherwise. As a result, the following holds. \( t_i(c) \geq t_{i-1}(c) \geq \ldots \geq t_0(c) \) and \( t_{i}(c) \geq t_{i}(c-1) \) where \( t_i \) is the time to open class \( i \). Therefore it is concluded that the revenue function \( R_H(.) \) is concave in \( c \) and that there is a unique optimal solution to this problem. In addition, \( \dot{R}_H(.) \) can be expressed in the following way.

\[
\dot{R}_H(t, c) = \sum_{i=1}^N \mu_i(t) \left[ F_i - \lambda(t, c) \right]^+
\]

where \( \left[ x \right]^+ = \max\{x, 0\} \).
The opening time of each bucket can be calculated in a double recursion complex: starting with \( c = 1 \) and ending with \( c = C \), and, at each \( c \), starting times for the classes are solved from the lowest class to the highest class. Please note that when we start at \( i = N \), we assume that \( F_{N+c} = 0 \), \( t_{N+c} = 0 \).

\[
F_i - F_{i+1} = \int_{t_{i+1}}^{t_i} d\tau \exp\left( \int_{t_{i+1}}^{\tau} d\tau' \sum_{j=1}^{i} \mu_j(\tau') \right) \left[ \sum_{k=1}^{i} (F_k - F_i) \mu_k(\tau) - \dot{R}_H(\tau, c - 1) \right]
\]

(4)

Please note that \( t_j \) is set to be \( T_0 \) when \( t_j \) is calculated to be greater than \( T_0 \) based on Equation (4). If \( t_j(c) = T_0 \), we must have \( t_i(c) = T_0 \ \forall i < j \).

The optimal expected revenue with available capacity \( c \) is, for \( t_i(c) \geq t \geq t_{i+1}(c) \),

\[
R_H(t, c) = F_{i+1} + R_H(t_{i+1}, c - 1) + \sum_{m=1}^{i} \int_{t_{i+1}}^{t} d\tau \exp\left( \int_{t_{i+1}}^{\tau} d\tau' \sum_{j=1}^{i} \mu_j(\tau') \right) \\
\cdot [F_m - F_{i+1} + R_H(\tau, c - 1) - R_H(t_{i+1}, c - 1)] \mu_m(\tau)
\]

(5)

After the revenue function is available for the period \([0, T_0]\), we can begin to solve for the starting times for the remaining capacity \( c+1 \). In addition, the bid price for each of these seats is given by

\[
\lambda(t, c) = F_{i+1} + R_H(t_{i+1}, c - 1) - R_H(t, c - 1) + \sum_{m=1}^{i} \int_{t_{i+1}}^{t} d\tau \exp\left( \int_{t_{i+1}}^{\tau} d\tau' \sum_{j=1}^{i} \mu_j(\tau') \right) \\
\cdot [F_m - F_{i+1} + R_H(\tau, c - 1) - R_H(t_{i+1}, c - 1)] \mu_m(\tau)
\]

(6)

where \( t_i \leq t \leq t_{i+1} \).

Our research extends this method to solve the period I problem.

**The Period I Problem**

During this period, aircraft assignments are uncertain. First we explain how to derive the salvage function \( R_f(T_0^+, c) \) before deriving the optimal policy for period I. To facilitate understanding, we tentatively assume that the starting capacity is \( C \).

*Salvage Function*
The salvage function $R_1(T^+_0, c)$ evaluates the revenue potential of the remaining capacity at time $T^+_0$. By adopting the result for period II, we assume the total revenue under the optimal policy after the aircraft assignment is fixed is $R_H(T^-_0, c)$ and is known at time $T_0^-$ as a function of $T_0$ and of available inventory $c$ where $(c_1 - c_2 \leq c \leq c_2)$.

The salvage function at time $T_0^+$ needs to consider two possible cases.

CASE A
The total capacity is determined to be $c_1$ at time $T_0$.

CASE B
The total capacity is determined to be $c_2$ at time $T_0$.

Each case has an associated probability $p$ and $1-p$ respectively. In Case A, the total available capacity at time $T_0^-$ becomes $c_1 + c - C$. Hence the maximum revenue in period II could only be $R_H(T^-_0, c_1 + c - C)$. In Case B, the total available capacity at time $T_0^-$ becomes $c_2 + c - C$. So, the maximum revenue in period II is $R_H(T^-_0, c_2 + c - C)$. Please note demand that occurs at time $T_0$ is realized at $T_0^-$; At all other times $t \neq T_0$, the demand is realized when it arrives.

Therefore the salvage function can be expressed in terms of the expected revenue as follows,

$$R_1(T^+_0, c) = pR_H(T^-_0, c_1 + c - C) + (1 - p)R_H(T^-_0, c_2 + c - C)$$  \hspace{1cm} (7)

Where $c$ is the remaining capacity.

Lemma I
If $G \geq F_1$, $R_1(T^+_0, c)$ is a monotonically increasing and concave function in $c$.

Proof
We know that $R_H(t, c)$ is a function that is monotonically increasing with the available capacity $c$ when $t \leq T_0^-$. Therefore, given a fixed assumed capacity $C$, at time $T$, $R_1(T^+_0, c)$ is a monotonically
increasing function of capacity $c$ available based on (7). In addition, when $t < T_0$, because $\lambda(t,1) \leq F_i$ and $\lambda(t,0) = G \geq F_i$. This leads to the following three results:

i) $\lambda(t,c) \leq \lambda(t,c-1) \forall c \geq 1$; ii) $\lambda(t,c) = \hat{\lambda}(t,c-1), \forall c \leq 0$; and iii) $\hat{\lambda}(t,1) \leq \hat{\lambda}(t,0)$. Hence, we obtain $\lambda(t,c) \leq \hat{\lambda}(t,c-1) \forall c$ and $R_i(t,c) - R_i(t,c-1) \leq R_i(t,c-1) - R_i(t,c-2)$ when $t \leq T_0$. Therefore $R_i(t,c)$ is a concave function in $c$ based on (7). We can conclude that $R_i(T_0^+, c)$ is a concave function in $c$ as well.

(End of proof)

**Determination of Assumed Starting Capacity**

According to the result for the Period II problem, the optimal policy is a function of the available capacity $c$. However, the capacity is not know with certainty prior to aircraft re-assignment. In the following, we discuss the assumption made about the initial capacity at time $T$. This is in line with the current practice that assumes a fixed capacity available on most flights.

By definition, the starting capacity for yield management at time $T$ must be within the range $[c_1, c_2]$. We must determine what starting capacity to assume within this range.

**Lemma II**

The optimal policy does not depend on the initial capacity assumed at time $T$.

**Proof**

Suppose there are two cases with different initial assumed capacities $c_1^*$ and $c_2^*$ where $(c_2 \geq c_2^* \geq c_1, c_1^* \leq c_1)$. The salvage function $R_i(T_0^+, c)$ for the two cases has a one to one mapping. Based on Equation (7), there exists the following:

$$R_i(T_0^+, c) \mid c_1^* = R_i(T_0^+, c + c_2^* - c_1^*) \mid c_2^*$$

(8)

where $R_i(T_0^+, c) \mid c^*$ represents the salvage function when initial capacity is assumed to be $c^*$ at time $T$.

That is, the salvage function with available capacity $c$ when $c_1^*$ is initially assumed is equal to the salvage function with an available capacity $c + c_2^* - c_1^*$ when $c_2^*$ is initially assumed. Suppose an optimal policy

$$\bar{\Theta} \mid c_1^* = \{(\bar{\theta}_1(t,c), \bar{\theta}_2(t,c),..., \bar{\theta}_n(t,c) : T_0 < t \leq T, c \leq c_1^*)\}$$

can lead to expected optimal
revenue $R^0_i$. We can always have a corresponding policy $\tilde{\theta}$ in the case when $c_2^*$ is initially assumed where

$$\tilde{\theta} \mid c_2^* = \{ (\tilde{\theta}_1(t,c), \tilde{\theta}_2(t,c), \ldots, \tilde{\theta}_n(t,c) : T_0 < t \leq T, c \leq c_2^* \}$$

with $\tilde{\theta}_i(t,c) = \theta_i(t,c + c_1^* - c_2^*)$. As a result, the same amount of revenue can be realized in two cases. In other words, the optimal control policy when a particular initial capacity is assumed can be mapped into a policy in which a different starting capacity is assumed. Therefore, the opening and closing of buckets does not depend on the initial capacity assumed.

The reason is that the boundary conditions are different. When $c_2$ is assumed at time $T$, the boundary condition is at $R_i(t,0)$; and the boundary condition is at $R_i(t, c_1 - c_2)$ when $c_1$ is assumed.

(End of proof)

For simplicity of expression, from this point on we assume that the initial starting capacity is $c_2$ unless otherwise specified.

Optimal Policy for the Period I Problem

Lemma III

During period I, the following conclusions developed by Liang (1999) still hold,

a) For the optimal policy $\Theta$, $\Theta_i(t,c) = 1$ if $F_i > \lambda(t,c)$ and $= 0$ if $F_i < \lambda(t,c)$

b) $d\lambda(t,c)/dt \geq 0$; for $c > 1, \lambda(t,c) \leq \lambda(t,c-1)$ $\forall t \in [T_0, T]$

Proof

The revenue function for the period I problem is as follows.

$$R^0_i(t,c) = \int_{T_0}^{t} d\tau \exp[-\int_{\tau}^{t} d\tau' \sum_{i=1}^{N} \mu_i^\theta(\tau', c)] \sum_{j=1}^{N} [F_j + R^0_j(\tau,c-1)] \mu_j^\theta(\tau,c)$$

$$+ \exp[-\int_{T_0}^{t} d\tau' \sum_{i=1}^{N} \mu_i^\theta(\tau', c)] R^0_i(T_0^+, c)$$

(9)

We provide a more general recursion. Please note that this expression reduces to the form presented in (1b) when $R^0_i(T_0^+, c)$ is equal to zero.
The derivative with respect to \( t \), which is the changing rate of expected revenue when no booking is accepted, has the same form as presented earlier in equation 2a.

\[
\dot{R}_i^o(t, c) = \sum_{j=1}^{N} \left[ F_j - \dot{R}_j^o(t, c) + R_j(t, c-1) \right] \mu_j^o(t, c)
\]  

(2a')

for \( t > T_0 \).

Therefore the proof for a) presented by Liang (1999) is also valid for this period I problem.

Now we discuss the proof for result b).

If we partially integrate equation (9) we have the following,

\[
\lambda(t, c) = \int_{T_0^+}^{t} \tau \exp(-\int_{\tau}^{t} \sum_{i=1}^{N} \mu_i^o(\tau', c) ) \left[ \sum_{j=1}^{N} F_j \mu_j^o - \dot{R}_i(\tau, c-1) \right] d\tau
\]  

+ \exp(-\int_{T_0^+}^{t} \sum_{i=1}^{N} \mu_i^o(\tau', c) ) \left[ R_i(T_0^+, c) - R_i(T_0^+, c-1) \right]

(10)

It is interesting to see that \( \lambda(T_0^+, c) = R_i(T_0^+, c) - R_i(T_0^+, c-1) \) has a decreasing effect on \( \dot{\lambda}(t, c) \) as \( t - T_0 \) increases. This can also be considered a generalized expression for the bid price.

Similar to Liang (1999), the following inequality holds,

\[
\lambda(t, c) - \lambda(t, c-1) \leq \\
-\int_{T_0}^{t} \tau \exp(-\int_{\tau}^{t} \sum_{i=1}^{N} \mu_i^o(\tau', c) ) \dot{\lambda}(t, c) d\tau
\]  

+ \exp(-\int_{T_0}^{t} \sum_{i=1}^{N} \mu_i^o(\tau', c) ) \left[ R_i(T_0^+, c) - R_i(T_0^+, c-1) \right]

- \left[ R_i(T_0^+, c-1) - R_i(T_0^+, c-2) \right] \leq \\
-\int_{T_0}^{t} \tau \exp(-\int_{\tau}^{t} \sum_{i=1}^{N} \mu_i^o(\tau', c) ) \dot{\lambda}(t, c) d\tau

(11)

Following the same steps as in Liang (1999), result b) can be proved valid.

(End of proof)
Lemma III determines the monotonic property of bid prices at different capacities available prior to time $T_0$. Therefore the Period II problem maintains the same solution structure as the Period I problem. In the following, we derive methods to decide the optimal opening time of each fare class and calculate the optimal revenue in the context of uncertain aircraft assignment.

We assume that at time $T$ the assumed capacity is $c_2$. Therefore the seat inventory available for sale (based on $c_2$) at any time $t$ ($t \geq T_0$) is never negative. The boundary condition can be expressed as follows.

$$R_j(T_0^+, 0) = -pG(c_2 - c_1)$$

(12)

Further, we obtain,

$$R_j(t, 0) = -pG(c_2 - c_1), \forall t > T_0$$

(13)

As we have mentioned above, there exists a unique optimal solution for the period I problem. We can apply the similar solution procedure to period I problem. The generalized revenue function results in changes in the solution process. We clarify these as follows.

The expected revenue function $R_j(t, c)$ in terms of the opening times for each fare class can be expressed as follows.

$$R_j(t_2, t_3, . . . , t_N; c)$$

$$= \sum_{i=1}^{N} \int_{T_0}^{t_i} d\tau \exp(-\sum_{j=1}^{N} H(t_j - \tau) \int_{\tau}^{t_j} \mu_j(\tau') d\tau')$$

$$\cdot [F_i + R_j(\tau, c - 1)] \mu_i(\tau)$$

$$+ \exp(-\sum_{j=1}^{N} H(t_j - T_0) \int_{T_0}^{t_j} \mu_j(\tau') d\tau') R_j(T_0^+, c)$$

(14)

Taking partial derivative with respect to $t_N$, the condition $\partial R(t_2, t_3, . . . , t_N)/\partial t_N = 0$ leads to

$$F_{t_N} + R_j(t_N, c - 1) =$$

$$\sum_{i=1}^{N} \int_{T_0}^{t_i} d\tau \exp(-\sum_{j=1}^{N} \int_{\tau}^{t_j} \mu_j(\tau') d\tau')(F_i + R_j(\tau, c - 1)) \mu_i(\tau)$$

$$+ \exp(-\sum_{j=1}^{N} \int_{T_0}^{t_j} \mu_j(\tau') d\tau') R_j(T_0^+, c)$$

(15)
Following the same method as in Liang (1999), we find

\[
F_N = \int_{t_N}^{t_h} d\tau \exp\left(\sum_{j=1}^{N} \int_{t_j}^{\tau} \mu_j(\tau') d\tau'\right) \left[ \sum_{i=1}^{N} (F_i - F_N) \mu_i(\tau) - R_i(T_0^+, c) - R_i(T_0^+, c - 1) \right] + R_i(T_0^+, c) - R_i(T_0^+, c - 1) \tag{16}
\]

At this point, we are able to solve for \( t_N \). If it turns out that \( t_N > T \), set \( t_i = T \) \( \forall i \). If, on the other hand, \( t_N < T \), solve \( t_{N-1} \) in a similar way as for \( t_N \) until the first bucket \( m \) for which \( t_m > T \). The solution is substituted back into Equation (14) and we have the same result as in Liang (1999). This is provided in equation (17).

\[
R_i(t_1, t_2, t_3, ... t_N; c) = F_N + R_i(t_N, c - 1)
+ \sum_{i=1}^{N-1} \int_{t_i}^{t_h} d\tau \exp\left(-\sum_{j=1}^{N-1} H(t_j - \tau) \int_{\tau}^{t_j} \mu_j(\tau') d\tau'\right) [F_i - F_N + R_i(\tau, c - 1) - R_i(t_N, c - 1)] \mu_i(\tau) \tag{17}
\]

It should now be clear that by repeating the same method, we can obtain the solution for \( i, i < m \), in the form shown in (4).

This is a double recursion process starting with \( c = 0 \) and \( i = N \) and progressing to higher buckets and larger capacities \( c \). If \( t_j(c) \geq T \), then all the variables \( t_i = T \) for \( i \leq j \).

In this way, we have extended Liang’s results to a more general case with a salvage function at the end of the planning horizon. When the salvage function is zero, our solution reduces to Liang’s. What is worthy of a mention is that the satisfying and simple form of the solution still generally holds in this context. The salvage function only affects calculation of the first bucket opened before \( T_0 \).

In the following section, we provide more results about yield management under possible aircraft reassignment.

**OPTIMAL YIELD MANAGEMENT WHEN CAPACITIES ARE SUBJECT TO CHANGE**

Here we compare our optimal control policy in which future capacity changes are anticipated to one without anticipation. We further provide a way to quantify the revenue improvement due to anticipating future changes. In addition, we present a method to generate an estimate (upper bound) on the optimal revenue.
Comparison of the Optimal Policy with the Case with Fixed Capacity Assignment

Proposition 1

Let $\lambda^*(t, c)$ represent the bid price in the case in which the capacity is always $c_1$ and let $\lambda(t, c)$ represent the bid price in the case the assignment of capacity at time $T_0$ is subject to an upgrade from $c_1$ to $c_2$. Then we have the following results assuming that in both cases the starting capacity is $c_1$:

$$\lambda^*(t, c) \geq \lambda(t, c) \quad \text{when} \quad t \geq T_0$$  \hspace{1cm} (18)

Proof

To make the two cases comparable, we assume that the same overbooking penalty $G$ is applied and that $G \geq F_1$.

We can easily find that,

$$\lambda(T_0^+, c) = p[R_H(T_0^-, c) - R_H(T_0^-, c - 1)] + (1 - p)[R_H(T_0^-, c + \Delta) - R_H(T_0^-, c + \Delta - 1)]$$  \hspace{1cm} (19)

Where $\Delta = c_2 - c_1$

In addition, we have the following,

$$\lambda^*(T_0^+, c) = R_H(T_0^-, c) - R_H(T_0^-, c - 1)$$  \hspace{1cm} (20)

Since $R_H(T_0^-, c)$ monotonically increases and is concave in $c$, the following holds.

$$\lambda^*(T_0^+, c) \geq \lambda(T_0^+, c)$$  \hspace{1cm} (21)

Now we show that $\lambda^*(t, c) \geq \lambda(t, c)$ when $t > T_0$.

(a) We first show that $\lambda^*(t, 0) \geq \lambda(t, 0)$ when $t > T_0$. 


We have \( \lambda(t, 0) \leq \lambda(t, c_1 - c_2), t > T_0 \). Further, based on (19), \( \lambda(t, c_1 - c_2) = G \). Therefore, \( \lambda(t, 0) \leq G \). On the other hand, \( \lambda^*(t, 0) = G \). Therefore, the result is proved.

(b) Next we prove that \( \lambda^*(t, c) \geq \lambda(t, c) \) given \( \lambda^+(t, c-1) \geq \lambda(t, c-1) \) when \( t > T_0, c > 0 \).

We know from (21) \( \lambda^+(T_0^+, c) \geq \lambda(T_0^+, c) \). Now first take the case \( \lambda^+(T_0^+, c) > \lambda(T_0^+, c) \). We know that \( \lambda(\cdot) \) is a continuous function of \( t \). There are two possibilities: one is that \( \lambda^+(t, c) > \lambda(t, c), \forall t > T_0 \), while the other is that \( \lambda^+(t, c) = \lambda(t, c) \exists t > T_0 \). In the latter case, we suppose at time \( t^* > T_0 \), there first exists \( \lambda^+(t^*, c) = \lambda(t^*, c) \) while \( \lambda^+(t, c) > \lambda(t, c) \) when \( T_0 < t < t^* \).

Again, since \( \lambda(t, c) = \sum_{i=1}^{N} \mu_i(t) \left[ P \left[ F_i - \lambda(t, c) \right] - P \left[ F_i - \lambda(t, c-1) \right] \right] \) based on (3),

\[
\lambda(t, c) - \lambda^+(t, c) = \sum_{i=1}^{N} \mu_i(t) \left[ P \left[ F_i - \lambda(t, c) \right] - P \left[ F_i - \lambda^+(t, c) \right] \right] \leq 0
\]

Therefore \( \lambda^+(t^*, c) \geq \lambda(t^*, c) \) and we can conclude that \( \lambda^+(t^* + dt, c) \geq \lambda(t^* + dt, c) \).

For the case where \( \lambda^+(T_0^+, c) = \lambda(T_0^+, c) \), we can reach the same result in a similar way. As a result, the inequality (18) always holds.

(End of proof)

In other words, in the case of a possible capacity upgrade, the bucket should be opened earlier than in the case without capacity upgrade. We think this is an interesting and important result.

Likewise, the result can be shown that the bucket should be opened later in the case of a possible capacity downgrade than in the case of no capacity downgrade. We can easily draw such a conclusion as follows.

**Proposition 2**

During the yield management process in the case of a possible capacity swap, the bucket is opened earlier than in the case with a constant smaller capacity and later than in the case with a constant larger capacity.
Revenue Improvement

In this section, we compare the revenue associated with the two cases.

Case I: The optimal policy is made in anticipation of future capacity change. The revenue in this case is $R_i(T, c_2)$.

Case II: The myopic optimal policy is made based on an a priori assignment of $c_1$ until the moment of aircraft swap at $T_0$ after which optimal policy is based on the new capacity realized. The revenue in this case is denoted $R^{MYOP}(T, c_1)$.

Here we calculate the revenue for Case II. In this case, there are two possibilities. One is that the capacity assigned turns out to be $c_1$ at a probability of $p$ with total revenue of $R_H(T, c_1)$. The other is that the capacity assigned turns out to be $c_2$ (capacity upgrade) at a probability of $1-p$ with total revenue $R^M(T, c_1)$.

Therefore the total revenue $R^{MYOP}(T, c_1)$ can be expressed as follows.

$$ R^{MYOP}(T, c_1) = (p)R_H(T, c_1) + (1-p)R^M(T, c_1) \tag{23} $$

This equation can be transformed as follows,

$$ R^{MYOP}(T, c_1) = R_H(T, c_1) + (1-p)[R^M(T, c_1) - R_H(T, c_1)] \tag{24} $$

We need to find out the difference between $R^M(T, c_1)$ and $R_H(T, c_1)$. Noting that the optimal policies and therefore expected revenue realized before $T_0$ are the same for $R^M(T, c_1)$ and $R_H(T, c_1)$, we have the following result,

$$ R^M(T, c_1) - R_H(T, c_1) = \sum_{c=1}^{n} P(T_0^+, c)[R_H(T_0^{-}, c + \Delta) - R_H(T_0^{-}, c)] \tag{25} $$

Where $P(T_0^+, c)$ is the probability of having $c$ seats left at time $T_0^+$ when a priori capacity $c_1$ is assumed.
Now we examine how to calculate $P(T_0^+, c)$. 

The probability of having $c_i$ seats available at time $T_0^+$ is equal to the probability of having no booking accepted, which can be calculated in the following form. Please note that we use $t_j(c)$ as the optimal opening time of bucket $i$ with a priori capacity $c_i$. 

$$P(T_0^+, c_i) = \exp \left( - \sum_{j=1}^{N} H(t_j(c_i) - T_0) \int_{T_0}^{t_j(c_i)} \mu_j(\tau')d\tau' \right)$$  \hspace{1cm} (26)$$

Furthermore, the probability of having $c_i - 1$ seats available at time $T_0^+$ is the probability that there was only one booking accepted, which is expressed as follows.

$$P(T_0^+, c_i - 1) = \sum_{i=1}^{N} \int_{T_0}^{t_j(c_i)} d\tau_i \exp \left( - \sum_{j=1}^{N} \int_{T_0}^{\min(t_j(c_i), \tau_i)} H(\tau' - T_0) \mu_j(\tau')d\tau' \right) \cdot \exp \left( - \sum_{j=1}^{N} H(t_j(c_i) - \tau_i) \int_{\tau_i}^{t_j(c_i)} \mu_j(\tau')d\tau' \right) H(\tau_i - T_0) \mu_i(\tau_i)$$  \hspace{1cm} (27)$$

In a similar way, we can find the probability of having a specific capacity available at time $T_0^+$ as follows,

$$P(T_0^+, c_i - n) = \prod_{k=1}^{n} \sum_{i=1}^{N} \int_{T_0}^{\min(t_j(c_i + 1 - k), \tau_i)} H(\tau_k - T_0) d\tau_k \cdot \exp \left( - \sum_{j=1}^{N} \int_{\tau_i}^{\min(t_j(c_i + 1 - k), \tau_i)} H(\tau' - \tau_k) \mu_j(\tau')d\tau' \right) \mu_i(\tau_k) \cdot \exp \left( - \sum_{j=1}^{N} \int_{T_0}^{\min(t_j(c_i), \tau_i)} H(\tau' - T_0) \mu_j(\tau')d\tau' \right)$$  \hspace{1cm} (28)$$

Where $\tau_i$ is the acceptance time of $i^{th}$ booking, $\tau_0 = +\infty$, and $n$ is the number of seats sold before $T_0$ ($1 \leq n \leq c_i$).

In addition, $R_f(T, c_1)$ and $R_H(T, c_1)$ can be calculated in the method explained before. Therefore, the total improvement of revenue equals to $R_f(T, c_2) - R^{MYOP}(T, c_1)$.
In a similar way, we can calculate the revenue improvement as opposed to the case in which yield management is based on an a priori capacity \( c_2 \) during the period before \( T_0 \) without anticipating the possible capacity downgrade.

\textit{An Upper Bound on the Optimal Revenue}

We further present a method for developing an upper bound on the optimal revenue.

\textbf{Lemma IV}

Suppose there are two salvage functions at time \( T_0 \): \( R_h(T_0, c) \) and \( R_f(T_0^+, c) \) under the assumption of some initial starting capacity \( C \). Assuming the booking request is the same as defined in the problem in both cases, the following result must hold

\[ R_h(T, C) \geq R_f(T, C) \]  \( (30) \)

if the following relationship exists,

\[ R_h(T_0, c) \geq R_f(T_0^+, c) \mid C, C - c_2 \leq c \leq C \]  \( (31) \)

\textbf{Proof}

The same policy applied to both cases leads to same probability of having the same residual capacity \( c \) at \( T_0 \) and the same revenue prior to \( T_0 \) since the demand prior to \( T_0 \) is the same in both cases. Therefore, the optimal policy with salvage function \( R_f(T_0^+, c) \mid C, C - c_2 \leq c \leq C \), leads to larger revenue when applied in the case with salvage function \( R_h(T_0, c), C - c_2 \leq c \leq C \). And, the optimal policy with salvage function \( R_h(T_0, c), C - c_2 \leq c \leq C \), leads to further improvement.

As a result, we can conclude that (30) holds.

(End of proof)

Based on Lemma IV, we have Proposition 3.

\textbf{Proposition 3}

The optimal policy anticipating the future capacity change leads to revenue bounded from above by \( R_h(T, (p)c_1 + (1-p)c_2) \).