Stochastic and Dynamic Airline Yield Management when Aircraft Assignments are Subject to Change

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ABSTRACT

In the commercial airline industry, aircraft assignments are subject to changes because of irregular operations and the practice of re-assigning planes to alternative routes due to unexpectedly high (or low) demands. Therefore there is a need for yield management techniques that explicitly incorporate the possibility of future capacity changes. Though there is a rich literature on the topic of yield management in the context of stochastic demand, no published research to date deals with the case in which capacity is also stochastic. In this paper, we propose a solution for the continuous stochastic dynamic yield management problem in which flight capacities are subject to change. This problem has wide practical applications in rental car and hotel management as well.

Key word: yield management, irregular airline operations
THE PROBLEM

Consider a booking process for a flight that starts at time $T$ before departure (at $t = 0$) in which there are $N$ distinct fare classes with fares $F_1, F_2, \ldots, F_n$ and in which we assume that $F_i > F_j$ for $i < j$. The corresponding competing booking processes of these classes are assumed to be independent inhomogeneous Poisson processes (no multiple seat bookings) with intensity functions $\mu_i(t), \ldots, \mu_n(t)$. These are sometimes referred to as incremental booking curves. It is assumed here that a request for a seat that has been denied will not be made later in another fare class. A yield management policy is one that determines when to open or shut down fare classes so as to maximize the expected revenue for each flight. At time $T_0$ ($T_0 > 0$), the aircraft could be reassigned, effectively changing the capacity of the flight. We assume that for any flight there are exactly two potential capacities, $c_1$ and $c_2$ and that $c_1 < c_2$. We further assume that these capacities have corresponding assignment probabilities $p$ and $(1-p)$. A yield management method that anticipates this potential future capacity change is needed.

In the airline industry, aircraft assignments are subject to changes due to fluctuations in demand. Because of practical considerations, aircraft changes are typically made about two weeks prior to departure ($T_0 = 14$ days). In general, assignments are fixed after that though mechanical failure or bad weather conditions often lead to last minute changes as well. Since irregular operations are typical in the airline industry, it is important to examine the impact of the irregularities in aircraft assignments on airline revenue. Therefore, an examination of methods to conduct optimal yield management in the context of stochastic assignment is needed. In this paper, we propose an optimal yield management policy for this problem. In addition, we address the following questions:

a) How should the decision to open or close fair buckets be made in the case of stochastic capacity?

b) How should penalty levels be set to avoid overbooking?

c) How does the optimal policy that anticipates future capacity changes compare with those without anticipation in terms of the control process and total revenue?

This problem has significant practical implications. The profit margins in the commercial airline industry are so thin that even a most modest improvement in total revenue could be critical to improving profitability. Related applications include rental car and hotel room management. In both of those applications, yield management must deal with uncertain future resources due to stochasticity of service durations.

RELATED LITERATURE

The previous decade has seen the evolution of yield management models from early static towards dynamic formulations. Static versions such as those presented in Belobaba (1,2), Brumelle et al (3), Curry (4) and Wollmer (5) do not consider the dynamics of the booking process. Later attempts realized that booking curve information could be very useful in that it allows a more accurate control of the seat inventory to capture additional revenue benefits and led to the development of dynamic programming models. Early dynamic models required the bookings to follow a lower class before a higher class pattern as seen in Brumelle and McGill (6). Lee and Hersh (7) relaxed this requirement of demand arrival order and built a discrete time dynamic programming model. Gallego and Van Ryzin (8), Papastavrou, Rajagopalan and Kleywegt (9) and Liang (10) study a dynamic yield management problem from slightly different angles. Gallego and Van Ryzin (8) assume a discrete time model. Papastavrou, Rajagopalan and Kleywegt (9) model the problem a knapsack problem with stochastic demand, which results in a yield management problem with booking curves for different classes having the same shape. Liang (10) specifically focuses on continuous dynamic yield management problem and provides optimal solution to this specific problem. However, none of them deals with the case of stochastic capacity in the future. All of them assume the inventory (capacity) available is deterministic while the demand is uncertain.
The problem we define here is similar to that studied in Liang (10) in which the yield management problem is formulated as a continuous time, stochastic dynamic programming model and in which an expression for the expected revenue in terms of the stochastic booking process and the control policies is derived. However, our work differs in that we consider the effect of possible aircraft capacity changes some time after the yield management process has begun. We extend the basic concepts and results from Liang (10) and develop solution method for this particular problem. Further, we discuss the impact of this stochastic capacity assignment on the optimal control policy and the total revenue.

In this paper, we decompose the problem into two stages and present solution techniques for these two stages separately. Our primary focus and main contributions concern the first stage that covers the time period prior to $T_0$. For this period, we begin by deriving the salvage function for the dynamic yield management problem. A salvage function evaluates the revenue potential of a remaining capacity at time $T_0$ at the moment before the capacity assignment is fixed. A discussion about how to determine the optimal starting capacity for yield management at time $T$ is presented. After introducing the optimal method to the period prior to aircraft reassignment, we present a condition to prevent overbooking followed by a comparison of the optimal control policy with the case without anticipation of aircraft reassignment. Revenue improvement as a result of the optimal policy in anticipation of aircraft reassignment is discussed as well.

**SOLUTION**

During the period after time $T_0$ (referred to as period II), the capacity is fixed. For the problem during this period, the yield management problem has already been solved successfully in Liang (10). We only need to examine the yield management strategy for the period before time $T_0$ in which aircraft capacities are uncertain. Without loss of generality, we assume that aircraft re-assignment is performed at time $T_0$. We denote the very moment before the aircraft reassignment as $T_0^-$, and the very moment after the reassignment $T_0^+$ (Please note that the + and – here refer to remaining time prior to and after the decision instant). To avoid notational confusion, $R_i(.)$ and $R_{ij}(.)$ are used as the revenue functions with and without capacity change respectively. For the period prior to $T_0$ (period I), we adopt the basic ideas presented in Liang (10) with $R_i(T_0^+, c)$ being the salvage function. The salvage function is derived under the optimal policy for period II, which will be explained in more detail later. The period I problem is a dynamic and stochastic yield management problem ending at time $T_0$. Therefore, an understanding of the method presented in Liang (10) helps with the understanding of the method proposed in this research. We provide his basic ideas here.

Let $\theta_i(t, c) = 1$ when class $i$ is open and $= 0$ when class $i$ is closed. The accepted booking processes are described by the intensities $\mu^\theta_i(t, c) = \mu_i(t)\theta_i(t, c)$, which are also inhomogeneous Poisson processes.

The probability of getting a booking in $dt$ at time $t$ in class $i$ is described as $\mu^\theta_i(t, c)dt$. The probability of getting the first booking in $dt$ at time $t$ in class $i$ is $\exp(- \int_0^t \sum_j \mu^\theta_j(\tau)d\tau)\mu^\theta_i(t)dt$. By first, Liang means first among all bookings, not just first in class $i$. This formula can be used to express optimal expected revenue inductively in capacity $c$. Once the first booking is accepted, there are $c-1$ seats remaining.

Now we present the solutions for periods II and I.
The Period II Problem

Let \( R_\theta(t, c) \) be the optimal expected revenue that can be realized from time \( t \) to \( 0 \) when there are \( c \) seats left at time \( t \). \( R_\theta(t, 0) = 0 \) when \( t < T_\theta \). The recursive expression for the expected revenue is as follows.

\[
R_\theta(t, c) = \int_0^t d\tau \exp \left[ \int_\tau^t d\tau' \sum_{j=1}^N \mu_j(\tau', c) \right] \sum_{j=1}^N \left[ F_j + R_\theta(\tau, c - 1) \right] \mu_j^\theta(\tau, c). 
\]  

(1)

The derivative with respect to \( t \) can be expressed as follows.

\[
\dot{R}_\theta(t, c) = \sum_{j=1}^N \left[ F_j - R_\theta(t, c) + R_\theta(t, c - 1) \right] \mu_j^\theta(t, c) 
\]  

(2)

The bid price can be expressed as follows. \( \lambda(t, c) \equiv R(t, c) - R(t, c - 1) \) and possesses the following property: \( d\lambda(t, c) / dt \geq 0, \forall t \in [0, T_\theta] \); for \( c > 1 \), \( \lambda(t, c) \leq \lambda(t, c - 1) \). The optimal control policy \( \Theta(t, c) = 1 \) if \( F \geq \lambda(t, c) \) and is 0 otherwise. As a result, the following holds. \( t(c) \geq t(c) \geq \ldots \leq t(c) \) and \( t(c) \geq t(c - 1) \) where \( t_i \) is the time to open class \( i \). Therefore it is concluded that the revenue function \( R_\theta(.) \) is concave in \( c \) and that there is a unique optimal solution to this problem. As a result, \( \dot{R}_\theta(.) \) can be expressed in the following way.

\[
\dot{R}_\theta(t, c) = \sum_i \mu_i(t)[P(F_i - \lambda(t, c))] 
\]  

(3)

Where \( P(x) \) is a Heaviside function which is equal to 1 when \( x > 0 \) and 0 otherwise.

The opening time of each bucket can be calculated in the following way recursively, starting at \( i = N \) with \( F_{N, t} = 0, t_{N+i} = 0 \).

\[
F_i - F_{i+1} = \int_0^t d\tau \exp \left[ \int_\tau^t d\tau' \sum_{j=1}^i \mu_j(\tau') \right] \sum_{k=1}^i (F_k - F_i) \mu_k(\tau) - \dot{R}_\theta(\tau, c - 1) 
\]  

(4)

The optimal expected revenue with available capacity \( c \) is, for \( t_i(c) \geq t \geq t_{i+1}(c) \),

\[
R_\theta(t, c) = R_{i+1}(t_{i+1}, c - 1) + \sum_{m=1}^i \int_0^t d\tau \exp \left[ - \int_\tau^t d\tau' \sum_{j=1}^i \mu_j(\tau') \right] \mu_m(\tau) 
\]  

\[
+ \left[ F_m - F_{i+1} + R_\theta(\tau, c - 1) - R_\theta(t_{i+1}, c - 1) \right] \mu_m(\tau) 
\]  

(5)

After the revenue function is available for the period \([0, T_\theta]\), we can begin to solve the starting times for the remaining capacity \( c+1 \). The bid price is

\[
\lambda(t, c) = R_{i+1}(t_{i+1}, c - 1) - R_\theta(t, c - 1) + \sum_{m=1}^i \int_0^t d\tau \exp \left[ - \int_\tau^t d\tau' \sum_{j=1}^i \mu_j(\tau') \right] \mu_m(\tau) 
\]  

\[
+ \left[ F_m - F_{i+1} + R_\theta(\tau, c - 1) - R_\theta(t_{i+1}, c - 1) \right] \mu_m(\tau) 
\]  

(6)

Therefore the entire solution process is a double recursion complex: starting with \( c = 1 \) and end with \( c = C \), and, at each \( c \), starting times for the classes are solved from the lowest class to the highest class. For more details, please refer to Liang (10).
Our research extends this method to solve the period I problem.

**The Period I Problem**

During this period, aircraft assignments are uncertain. First we explain how to derive the salvage function $R_I(T_0^+, c)$. To facilitate understanding, we tentatively assume that the starting capacity is $C$.

**Salvage Function**

Before deriving the optimal policy for period I, it is necessary to identify the salvage function $R_I(T_0^+, c)$ that evaluates the revenue potential of the remaining capacity $c$ at time $T_0^+$. By adopting the result for period II, we assume the total revenue $R_{II}(T_0^-, c)$ is known at time $T_0^-$ as a function of time $T_0$ and available inventory $c$ ($0 < c < C$) under the optimal policy after the aircraft assignment is fixed. To be more general, we extend the domain of $c$ to include negative values. Here we assume $R_{II}(T_0^-, c) = Gc$ when $c \leq 0$ where $G$ is the penalty for each seat overbooked.

The salvage function at time $T_0^+$ needs to consider two possible cases.

**CASE A**
The total capacity is determined to be $c_1$ at time $T_0$.

**CASE B**
The total capacity is determined to be $c_2$ at time $T_0$.

Each case is associated with a probability $p$ and $1-p$ respectively. In Case A, the total available capacity at time $T_0^-$ becomes $c_1 + c - C$. Hence the maximum revenue in period II could only be $R_{II}(T_0^-, c \mid c_1) = R_{II}(T_0^-, c_1 + c - C)$. In Case B, the total available capacity at time $T_0^-$ becomes $c_2 + c - C$. So, the maximum revenue in period II would be $R_{II}(T_0^-, c \mid c_2) = R_{II}(T_0^-, c_2 + c - C)$. Please note that, for convenience, we assume there is no demand arrival between time $T_0^-$ and $T_0^+$. Demand that occurs at time $T_0$ happens at $T_0^-$. At all other times $t \neq T_0$, the demand arrives at time $t$.

Therefore the salvage function can be expressed as follows,

$$R_I(T_0^+, c) = p R_{II}(T_0^-, c_1 + c - C) + (1 - p) R_{II}(T_0^-, c_2 + c - C) \tag{7}$$

Where $c$ is the remaining capacity.

As we know, $R_{II}(t, c)$ is a monotonically increasing function with the capacity available $c$ when $t \leq T_0^-$. Therefore, given a fixed capacity $C$ to assume at time $T$, $R_I(T_0^+, c)$ is a monotonically increasing function of capacity $c$ available. In addition, since $\lambda(t, c) \leq \lambda(t, c - 1)$, we have $R_{II}(t, c) - R_{II}(t, c - 1) \leq R_{II}(t, c - 1) - R_{II}(t, c - 2)$ when $t \leq T_0$. Therefore $R_{II}(t, c)$ is a concave
function in $c$. We can easily conclude that $R_i(T_0^+, c)$ is a concave function in $c$ as well. However, it is easily seen that given $c$, the larger the $C$ is, the smaller the salvage value is.

**Determination of Assumed Starting Capacity**

We take it as self-evident here that the starting capacity for yield management at time $T$ must be within the range $[c_1, c_2]$. Within this range, we must determine what starting capacity should be assumed.

**Observation I**

The optimal policy does not depend on the initial capacity assumed at time $T$.

**Proof**

Suppose there are two cases with different initial capacities assumed $c_1^*$ and $c_2^*$ ($c_2^* \geq c_1^* \geq c_1 \geq c_1$). The salvage function $R_i(T_0^+, c)$ for the two cases has a one to one mapping. There exists the following:

$$R_i(T_0^+, c) | c_1^* = R_i(T_0^+, c + c_2^* - c_1^*) | c_2^* \tag{8}$$

That is, the salvage function with available capacity $c$ when $c_1^*$ is initially assumed equals to the salvage function with an available capacity $c + c_2^* - c_1^*$ when $c_2^*$ is initially assumed. Suppose the optimal policy when $c_1^*$ is assumed dictates the opening (closing) of bucket $F_i$ at a time $t$ ($t > T_0$) when remaining capacity $c$ is available. This control policy can also be adopted in the case $c_2^*$ at the time $t$ and available capacity $c - c_1^* + c_2^*$. As a result, the same amount of revenue can be realized in two cases. In other words, the optimal control policy when one initial capacity is assumed can be mapped into another case where a different starting capacity is assumed. Therefore, opening and closing a bucket does not depend on the initial capacity assumed.

(End of proof)

The reason is that the boundary conditions for the two cases are different. When $c_2$ is assumed at time $T$, the boundary condition is at $R_i(t, 0)$; and the boundary condition is $R_i(t, c_1 - c_2)$ when $c_1$ is assumed. For simplicity, $c_2$ is assumed hereafter.

**Observation II**

There is a unique optimal policy for this dynamic stochastic yield management problem with uncertain capacity.

**Proof**

First we must demonstrate that $\lambda(t, c) \leq \lambda(t, c - 1)$.
Examining the functions $\mathcal{R}_H(T_0^-, c | c_1)$ and $\mathcal{R}_H(T_0^-, c | c_2)$, we observe that both are concave since

$$\lambda(t, c) \leq \lambda(t, c - 1)$$

when $t \leq T_0^-$, which means that

$$\mathcal{R}_H(T_0^-, c | c_1) - \mathcal{R}_H(T_0^-, c - 1 | c_1) \leq \mathcal{R}_H(T_0^-, c - 1 | c_1) - \mathcal{R}_H(T_0^-, c - 2 | c_1).$$

Therefore, the salvage function $\mathcal{R}_i(T_0^+, c)$ is also concave in $c$ (where $c$ is limited to integer values). From the equation below,

$$\mathcal{R}_i^\theta(t, c) = \int_{t_0}^t e^{-\int_{t_0}^r \sum_{i=1}^N p_i^\theta(r, c)} \sum_{j=1}^N \left[ F_j + \mathcal{R}_i^\theta(r, c - 1) \right] \mu_j^\theta(r, c)$$

$$+ e^{-\int_{t_0}^r \sum_{i=1}^N p_i^\theta(r, c)} \mathcal{R}_i^\theta(T_0^+, c)$$

we can conclude that the function $\mathcal{R}_i^\theta(t, c)$ is concave in $c$ when $t \geq T_0^-$ since it is a combination of an infinite number of concave functions. This means that $\lambda(t, c) \leq \lambda(t, c - 1)$ when $t \geq T_0^-$.

Now we show that $d\lambda(t, c) / dt \geq 0$.

Based on the result from Liang (10), we have the following,

$$\dot{\lambda}(t, c) = \sum_i \mu_i(t, c) \{ P[F_i - \lambda(t, c)] - P[F_i - \lambda(t, c - 1)] \}$$

Since $\lambda(t, c) \leq \lambda(t, c - 1)$ when $t \geq T_0^-$, we can easily have $\dot{\lambda}(t, c) \geq 0$ when $t \geq T_0^-$.

(End of proof)

*Optimal Policy for the Period II Problem*

We assume that the capacity to assume at time $T$ is $c_2$. Therefore the seat inventory available for sale during the time $t (t \geq T_0)$ is never negative. Therefore the boundary condition can be expressed as follows.

$$\mathcal{R}_i(T_0^+, 0) = -(1 - p)G(c_2 - c_1)$$

and further we have,

$$\mathcal{R}_i(t, 0) = -(1 - p)G(c_2 - c_1), \text{ for } t \geq T_0$$

As we have mentioned above, there exists a unique optimal solution to the period I problem. We can apply the same solution procedure for period II problem to period I problem. The only difference is, instead of optimizing over the interval between 0 and $T_0$, the period II problem is managed over the interval between $T_0$ and $T$.

In the following section, we provide more results about yield management under possible aircraft reassignment.
A Method to Prevent Overbooking

Overbooking is a serious concern in airlines industry. If cancellation of a booking is not allowed, overbooking should be prevented. In the case of an uncertain capacity in the future, there is a possibility of overbooking when the capacity assigned turns out to be smaller than expected. A necessary and sufficient condition to prevent over-booking is given as follows.

Observation III

When the remaining capacity is \( c \) (\( c \leq c_2 - c_1 \)) at a certain time point given the assumed starting capacity is \( c_2 \), the optimal policy is to close all buckets till time \( T_0^* \) if and only if the following condition is satisfied:

\[
F_1 - pG - (1 - p)[R_N(T_0^*, c_2 - c_1) - R_N(T_0^*, c_2 - c_1 - 1)] \leq 0
\]  

(12)

Proof

First we prove sufficiency of the condition.

It is self-evident that the optimal policy for the case when the remaining capacity is zero is to close all the buckets. Now suppose for the case when there are \( c-1 \) available at a time \( t > T_0^* \), where \( c \leq c_2 - c_1 \) the optimal policy is to close all buckets when the following condition is satisfied:

\[
F_1 - pG - (1 - p)\lambda(T_0^*, c) \leq 0
\]

This means that \( R_f(T_0^*, c - 1) = R_f(T_0^*, c - 1) \). Then we prove the same policy applies to the case when there are \( c \) seats available.

Considering the fact that

\[
\int_{T_0^*}^{t} d\tau \exp\left[-\int_{T_0^*}^{\tau} \sum_{i=1}^{N} \mu_i^0(\tau', c) \right] \sum_{j=1}^{N} \mu_j^0(\tau', c) \right] = 1
\]  

(13a)

and therefore

\[
R_f(T_0^*, c) = \int_{T_0^*}^{t} d\tau \exp\left[-\int_{T_0^*}^{\tau} \sum_{i=1}^{N} \mu_i^0(\tau', c) \right] \sum_{j=1}^{N} R_f(T_0^+, c) \mu_j^0 \exp\left[-\int_{T_0^*}^{\tau} \sum_{i=1}^{N} \mu_i^0(\tau', c) \right] \right] R_f(T_0^+, c)
\]  

(13b)

Again, we have

\[
R_f(t, c) = \int_{T_0^*}^{t} d\tau \exp\left[-\int_{T_0^*}^{\tau} \sum_{i=1}^{N} \mu_i^0(\tau', c) \right] \sum_{j=1}^{N} [F_j + R_f(\tau, c - 1)] \mu_j^0 \exp\left[-\int_{T_0^*}^{\tau} \sum_{i=1}^{N} \mu_i^0(\tau', c) \right] \right] R_f(T_0^+, c)
\]

The following holds,
\[
R_i(t, c) - R_i(T_0^+, c) = \int_{t_0}^{t} d\tau \exp\left[-\int_{\tau_0}^{\tau} d\tau' \sum_{i=1}^{N} \mu_i^O(\tau', c)\right] \sum_{j=1}^{N} [F_j + R_i(\tau, c-1) - R_i(T_0^+, c)] \mu_j^O \\
= \int_{t_0}^{t} d\tau \exp\left[-\int_{\tau_0}^{\tau} d\tau' \sum_{i=1}^{N} \mu_i^O(\tau', c)\right] \sum_{j=1}^{N} [F_j + R_i(T_0^+, c-1) - R_i(T_0^+, c)] \mu_j^O \\
\leq \int_{t_0}^{t} d\tau \exp\left[-\int_{\tau_0}^{\tau} d\tau' \sum_{i=1}^{N} \mu_i^O(\tau', c)\right] \sum_{j=1}^{N} [F_j - pG + (1 - p)(R_{\mu}(T_0^+, c-1) - R_{\mu}(T_0^-, c))] \mu_j^O \\
= 0
\]

(13c)

Since there clearly exists \( R_i(t, c) - R_i(T_0^+, c) \geq 0 \) we must have \( R_i(t, c) = R_i(T_0^+, c) \) and therefore, \( \mu_j^O(\tau, c) = 0, \forall j \), which means that all the buckets are closed at \( c \) at time \( t > T_0 \) where \( c \leq c_2 - c_1 \).

Now we prove the necessity of the condition.

When the optimal policy dictates closing of all the buckets at time \( t > T_0 \) with capacity \( c (c \leq c_2 - c_1) \), there holds \( R_i(t, c) = R_i(T_0^+, c) \) where \( t > T_0 \). In addition, all the buckets must be closed under optimal policy at capacity \( (c - 1) \) as well. This means \( R_i(t, c-1) = R_i(T_0^+, c-1) \) where \( t > T_0 \).

Suppose the condition does not exist. That is,
\[
F_i - pG - (1 - p)(R_{\mu}(T_0^+, c) - R_{\mu}(T_0^-, c)) > 0
\]

(14)

Similar to (13), we then have the following,
\[
R_i(t, c) - R_i(T_0^+, c) = \int_{t_0}^{t} d\tau \exp\left[-\int_{\tau_0}^{\tau} d\tau' \sum_{i=1}^{N} \mu_i^O(\tau', c)\right] \sum_{j=1}^{N} [F_j + R_i(\tau, c-1) - R_i(T_0^+, c)] \mu_j^O \\
= \int_{t_0}^{t} d\tau \exp\left[-\int_{\tau_0}^{\tau} d\tau' \sum_{i=1}^{N} \mu_i^O(\tau', c)\right] \sum_{j=1}^{N} [F_j + R_i(T_0^+, c-1) - R_i(T_0^+, c)] \mu_j^O \\
= \int_{t_0}^{t} d\tau \exp\left[-\int_{\tau_0}^{\tau} d\tau' \sum_{i=1}^{N} \mu_i^O(\tau', c)\right] \sum_{j=1}^{N} [F_j - pG + (1 - p)(R_{\mu}(T_0^+, c-1) - R_{\mu}(T_0^-, c))] \mu_j^O \\
+ \int_{t_0}^{t} d\tau \exp\left[-\int_{\tau_0}^{\tau} d\tau' \sum_{i=1}^{N} \mu_i^O(\tau', c)\right] [F_j - pG + (1 - p)(R_{\mu}(T_0^+, c-1) - R_{\mu}(T_0^-, c))] \mu_j^O
\]

(15)

Then we can have a policy \( \theta_i^* = 1 \), and \( \theta_i^* = 0, \forall i \neq 1 \) such that
\[ R_i(t, c) - R_i(T_0^+, c) > 0 \]

which violates the original assumption. This means when an optimal policy dictates closing of all the buckets at capacity available \( c \ (c \leq c_2 - c_1) \), inequality (14) does not hold and therefore condition (12) must be satisfied.

(End of proof)

Therefore, if the penalty \( G \) of having one unit negative capacity is so large as specified above, there is no reason to accept a booking when seat inventory available is smaller than \( c_2 - c_1 \). This observation helps set the right penalty level to prevent overbooking when necessary. Of course, this result can be generalized as follows.

When the remaining capacity is \( c \ (c \leq c_2 - c_1) \) at a certain time point \( t \) given the assumed starting capacity is \( c_2 \), the optimal policy is to close bucket \( i \) till time \( T_0 \) if and only if the following condition is satisfied:

\[ F_i - pG - (1 - p)[R_H(T_0^+, c_2 - c_1) - R_H(T_0^-, c_2 - c_1 - 1)] \leq 0 \]

Proof for this generalized result is similar to that of Observation III, and therefore is skipped here.

**Optimal Yield Management When Capacities are Subject to Change**

Here we discuss the optimal policy and its relation to the case without capacity change at time \( T_0 \).

**Proposition**

Let \( \lambda^*(t, c) \) represent the bid price in the case the assignment of capacity is always \( c_1 \) and \( \lambda(t, c) \) bid price in the case the assignment of capacity at time \( T_0 \) is subject to an upgrade from \( c_1 \) to \( c_2 \). Then we have the following observation under the condition that the starting capacity is supposed to be \( c_1 \) and no overbooking is allowed.

\[ \lambda^*(t, c) \geq \lambda(t, c) \text{ when } t \geq T_0 \]  

(16)

Proof

We can easily find that,

\[ \lambda(T_0^+, c) = p[R_H(T_0^+, c) - R_H(T_0^-, c - 1)] + (1 - p)[R_H(T_0^-, c + \Delta) - R_H(T_0^-, c + \Delta - 1)] \]

Where \( \Delta = c_2 - c_1 \)

(17)

In addition, we have the following,
\( \lambda^*(T_0^+, c) = R_{\eta}(T_0^-, c) - R_{\eta}(T_0^-, c - 1) \)  

(18)

Since \( R_{\eta}(T_0^-, c) \) monotonically increases and is concave in \( c \), the following holds.

\[
\hat{\lambda}(T_0^+, c) \geq \lambda(T_0^+, c)
\]

(19)

Now let's show that \( \hat{\lambda}(t, c) \geq \lambda(t, c) \) when \( t \geq T_0 \)

(a) we first show that \( \hat{\lambda}^*(t, 0) \geq \lambda(t, 0) \) when \( t \geq T_0 \)

We know that \( \hat{\lambda}(t, 0) = G, t > T_0 \).

\[
\lambda(T_0^+, 0) = R_{\lambda}(T_0^+, 0) - R_{\lambda}(T_0^+, -1) = pG + (1 - p)[R_{\eta}(T_0^-, c_2 - c_1) - R_{\eta}(T_0^-, c_2 - c_1 - 1)] \leq G
\]

(20)

It is self-evident that the following holds.

\[
\hat{R}_{\lambda}^\Theta(t, 0) = \sum_{j=1}^{N}[F_j - R_{\lambda}^\Theta(t, 0) + R_{\lambda}(t, -1)]\mu_j^\Theta(t, 0) = 0
\]

(21)

Since when \( c \leq 0 \), the optimal policy is to set \( \mu_j^\Theta = 0 \) \( \forall j \). Therefore,

\[
\hat{\lambda}(t, c) = \hat{R}(t, c) - \hat{R}(t, c - 1) = 0 \text{ when } t \geq T_0, c \leq 0
\]

(22)

Together with (20), we have \( \lambda(t, 0) \leq G \)

Therefore \( \lambda^*(t, 0) \geq \lambda(t, 0) \) when \( t \geq T_0 \)

(b) Then we prove that \( \lambda^*(t, c) \geq \lambda(t, c) \) given

\[
\hat{\lambda}(t, c - 1) \geq \lambda(t, c - 1) \text{ when } t \geq T_0, c \geq 0
\]

Again, \( \lambda(.) \) is a continuous function of \( t \). And we know \( \lambda^*(T_0^+, c) \geq \lambda(T_0^+, c) \). Now suppose at the time \( t^* > T_0 \) there first exists \( \hat{\lambda}(t^*, c) = \lambda(t^*, c) \) while \( \hat{\lambda}(t^*, c) > \lambda(t^*, c) \) when \( T_0 < t < t^* \).

Again, since \( \lambda(t, c) = \sum_i \mu_i(t, c)\{P[F_i - \lambda(t, c)] - P[F_i - \lambda(t, c - 1)]\}, \)

\[
\hat{\lambda}(t^*, c) - \hat{\lambda}(t^*, c) = \sum_i \mu_i(t^*, c)\{P[F_i - \lambda(t^*, c - 1)] - P[F_i - \lambda(t^*, c - 1)]\} \leq 0
\]
Therefore $\lambda^* (t^*, c) \geq \lambda(t^*, c)$ and we can conclude that $\lambda^* (t^* + dt, c) \geq \lambda(t^* + dt, c).$ As a result, the following inequality always holds,

$$\lambda^*(t, c) \geq \lambda(t, c), t \geq T_0$$

In other words, in the case of possible capacity upgrade, the bucket should be opened earlier than in the case without capacity upgrade.

Likewise, the result can be shown that the bucket should be opened later in the case of a possible capacity downgrade than in the case of no capacity downgrade. We can easily draw such a conclusion as follows.

**Proposition**

During the yield management process in the case of a possible capacity swap, the bucket is opened earlier than in the case with a constant smaller capacity and later than in the case with a constant bigger capacity.

**Revenue Improvement**

In this section, comparisons will be made in terms of revenue between two cases.

Case I: The optimal policy is made in anticipation of future capacity change. The revenue in this case is $R^*_i(T, c_2)$

Case II: Myopic optimal policy is made based on an a priori assignment of $c_1$ until the moment of aircraft swap at $T_0$ after which optimal policy is based on the new capacity realized.

Here we calculate the revenue in Case II. In this case, there are two possibilities. One is that the capacity assigned turns out to be $c_1$ with probability $p$ and total revenue $R^M(T, c_1)$. The other is that the capacity turns out to be $c_2$ (capacity upgrade) with probability $1-p$ and total revenue $R^M(T, c_1)$.

Therefore the total revenue $R^*(T, c_1)$ in the second case can be expressed as follows.

$$R^*(T, c_1) = pR^*_i(T, c_1) + (1-p)R^M(T, c_1)$$

Now we need to calculate the revenue $R^M(T, c_1)$.

In light of the method for provided for the period II problem, we can easily obtain the revenue function $R^*_i(T_0^-, c)$ for all $c \leq c_2$. Therefore the following holds.

$$R^M(T_0^+, c) = R^*_i(T_0^-, c + \Delta), \text{ where } \Delta = c_2 - c_1$$
Using the method for period I problem, we can easily obtain the revenue $R^M(T, c_1)$. Therefore, the total improvement of revenue equals to $R_f(T, c_{1}) - R^*(T, c_{2})$.

In the same way, we can calculate the revenue improvement as opposed to the case in which yield management is based on an a priori capacity $c_2$ before $T_0$.

CONCLUSION

In this paper, a dynamic and stochastic yield management problem when capacity is subject to change is studied. A solution method is proposed. This problem is decomposed into a two-stage yield management problem. The result from second stage (period II) is used to derive the salvage function for the first stage (period I) for determination of the optimal policy. The problem of how to determine the initial capacity to assume is also discussed. We specifically provide a sufficient and necessary condition for determination of the penalty to avoid overbooking. In addition a comparison is made in terms of both seat inventory control and total revenue between the optimal yield management policy for this problem and for the myopic one based on a priori assignment.

Though only the simple case of changing capacity is considered, the method developed here can be extended to a more general case where the capacity can be changed at multiple times and to multiple levels. In addition, future research opportunities on this problem are extensive. These include but are not limited to further examination of the extent to which the total revenue is affected by the time selected to make the changes in assignments, the affects of different higher or lower change probabilities. We hope to extend these results in the near future.

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