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Solving the bicriteria traffic equilibrium problem with variable demand and nonlinear path costs

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ABSTRACT

In this paper, we present an algorithm for solving the bicriteria traffic equilibrium problem with variable demand and nonlinear path costs. The path cost function considered is comprised of two attributes, travel time and toll, that are combined into a nonlinear generalized cost. Travel demand is determined endogenously according to a travel disutility function. Travelers choose routes with the minimum overall generalized costs. The algorithm involves two components: a bicriteria shortest path routine to implicitly generate the set of non-dominated paths and a projection and contraction method to solve the nonlinear complementarity problem (NCP) describing the traffic equilibrium problem. Numerical experiments are conducted to demonstrate the feasibility of the algorithm to this class of traffic equilibrium problems.

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1. Introduction

It is generally accepted that travelers consider a number of criteria (e.g., time, money, distance, safety, route complexity, etc.) when selecting routes. Presumably, these criteria are then combined in some manner to form a generalized cost for each particular route or path under consideration, and a route selected based on minimization of the generalized cost of the trip. Most commonly, it is assumed that travelers select the '*best*' route based on either a single criterion, such as travel time, or several criteria using a linear (or additive) path cost function. The linearity assumption offers the advantage that the traffic equilibrium problem can be solved without the need to store paths, which is a significant benefit, since it allows solution of large-scale network problems for which path enumeration is practically infeasible. However, as pointed out by Gabriel and Bernstein [1], there are many situations in which the linear path cost function is inadequate for addressing factors affecting a variety of transportation policies. Such factors include:

(i) Nonlinear valuation of travel time – small amounts of time are valued proportionately less than larger amounts of time.
(ii) Emissions fees – emissions of hydrocarbons and carbon monoxide are a nonlinear function of travel times.
(iii) Path-specific tolls and fares – most existing fare and toll pricing structures are not directly proportional to either travel time or distance.

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These, and other such factors, are generally difficult to accommodate without explicitly using path flows in the formulation and solution, particularly for traffic equilibrium problems involving multi-dimensional nonlinear path costs.

Despite the obvious usefulness of incorporating multiple criteria and relaxing the assumption of linear path costs for an important class of traffic equilibrium problems, there have been relatively few attempts to incorporate multiple criteria within route choice modeling. Recently, Dial [2,3] formulated a bicriteria user equilibrium assignment model based on out-of-pocket costs and travel time using a linear generalized path cost, and provided efficient algorithms for solving practical problems in planning applications. Blue et al. [4] proposed an algorithm for the bicriteria shortest path problem that considers two criteria: travel time and route complexity, represented by turning maneuvers. The algorithm uses a simple weighting method and assumes that all members of a particular user class use the same value of weight. Nagurney [5] and Nagurney and Dong [6] developed a multiclass, multicriteria traffic equilibrium model for fixed and elastic demands in which travelers for a class perceive their generalized cost on a route as a weighting of travel time and travel cost, where the weights are not only class-dependent but also link-dependent. Under the assumption that the nonlinear path cost function is known *a priori*, Scott and Bernstein [7] solved a constrained shortest path problem (CSPP) to generate a set of Pareto optimal paths and then identify the best path by evaluating the cost values of the alternative paths. In a later study, Scott and Bernstein [8] embedded the CSPP into the gradient projection method to solve the non-additive traffic equilibrium problem. Mixed results led the authors to conclude that the diagonalized subproblem was a poor approximation for the non-additive problem. Using a new gap function recently proposed by Facchinei and Soares [9], Lo and Chen [10] reformulated the non-additive traffic equilibrium problem as an equivalent unconstrained optimization and solved a special case involving fixed demand and route-specific costs. Chen et al. [11] provided a projection and contraction algorithm for solving the elastic traffic equilibrium problem with route-specific costs. Recently, some formulations and properties of the non-additive traffic equilibrium models were also explored, such as the nonlinear time/money relation [12], the uniqueness and convexity of the bicriteria traffic equilibrium problem [13], and the monotonicity of the mixed complementarity problem formulation [14]. Furthermore, Altman and Wynter [15] discussed the non-additive cost structures in both transportation and telecommunication networks.

In this paper, we consider the traffic equilibrium problem with variable demand, fixed tolls, and a nonlinear path cost function. We first discuss the bicriteria traffic equilibrium problem and its equivalent nonlinear complementarity formulation, and present the associated bicriteria shortest path problem (BCSSP) and solution algorithm. We then explore a class of projection and contraction (PC) methods developed by He [16] to solve the nonlinear complementarity problem (NCP) that characterizes this class of traffic equilibrium problem. The PC method is simple and can handle a general monotone mapping. Unlike the non-smooth equations/sequential quadratic programming (NE/SQP) method proposed by Gabriel and Bernstein [1] to solve the non-additive traffic equilibrium problem, the PC method does not require the mapping to be differentiable. It only assumes a monotone condition on the mapping. It uses three fundamental inequalities to construct the search direction and a self-adaptive scaling scheme to ensure convergence without the need to assume that the mapping satisfies the Lipschitz condition. For the bicriteria shortest path problem with nonlinear path costs, we use an exact method by Hansen [17] to automatically generate paths as needed. For purposes of illustration, we apply the combined BCSSP and PC algorithm to two networks and make comparisons with two linear path cost models.

2. The bicriteria traffic equilibrium problem and its equivalent nonlinear complementarity formulation

Consider a strongly connected network $[N,A]$, where N and A denote the sets of nodes and arcs, respectively. Let R and S denote subsets of N, for which travel demand q^{rs} is generated from origin $r \in R$ to destination $s \in S$. The independent variables are a set of path flows, denoted as f_p^{rs}, that must satisfy

$$\sum_{p\in P^{rs}} f_p^{rs} = q^{rs}, \quad \forall r \in R,\ s \in S, \tag{1}$$

where P^{rs} is a set of simple paths connecting r to s. Further, all path flows are restricted to be non-negative to ensure a meaningful solution, that is,

$$f_p^{rs} \geqslant 0, \quad \forall r \in R,\ s \in S,\ p \in P^{rs}. \tag{2}$$

Let v_a denote the traffic flow on link a. Then, the total flow on link a is simply the sum of all paths using that link

$$v_a = \sum_{r\in R}\sum_{s\in S}\sum_{p\in P^{rs}} f_p^{rs}\delta_{pa}^{rs}, \quad \forall a \in A, \tag{3}$$

where $\delta_{pa}^{rs} = 1$ if link a is on path p connecting r and s, and 0, otherwise.

A typical link cost function incorporating congestion effects expresses the travel time along the link as a function of the total link flow, that is,

$$t_a = t_a(v_a), \quad \forall a \in A. \tag{4}$$

For the single-criterion traffic equilibrium problem, the path cost is then simply the sum of the link travel times

$$\eta_p^{rs} = \sum_{a \in A} \delta_{pa}^{rs} t_a, \quad \forall r \in R,\ s \in S,\ p \in P^{rs}. \tag{5}$$

For the bicriteria traffic equilibrium problem with linear path costs based on travel time and toll, the generalized path cost can be obtained by a linear combination of the two criteria as follows:

$$\eta_p^{rs} = \alpha \sum_{a \in A} \delta_{pa}^{rs} t_a + \sum_{a \in A} \delta_{pa}^{rs} \tau_a, \tag{6}$$

where α is a "*value-of-time*" parameter (i.e., the amount that a traveler would be willing to pay in order to save time) and τ_a is the toll on link a.

The linearity assumption in (6) allows the traffic equilibrium problem to be formulated as a mathematical program that can be solved without the need to store paths (see Sheffi [18] for details). As pointed out by Gabriel and Bernstein [1], this assumption is rather restrictive and cannot adequately model certain important applications. For example, Hensher and Truong [19] showed that the valuation of travel time savings is nonlinear rather than linear. That is, small amounts of time are valued proportionally less than larger amounts of time. A possible nonlinear path cost function can be the following form:

$$\eta_p^{rs} = g_p\left(\sum_{a \in A} \delta_{pa}^{rs} t_a\right) + \sum_{a \in A} \delta_{pa}^{rs} \tau_a, \quad \forall r \in R,\ s \in S,\ p \in P^{rs}, \tag{7}$$

where g_p is a nonlinear function describing the value-of-time for path p. For this situation, the traffic equilibrium problem can only be formulated and solved in the path-flow domain.

For the associated travel demand, we assume variable demand with known travel disutility functions [20]. For each OD pair (r,s), there is a travel disutility π^{rs} given as a function of travel demand q (a vector of $\{\ldots,q^{rs},\ldots\}$), i.e.,

$$\pi^{rs} = \pi^{rs}(q), \quad \forall r \in R,\ s \in S. \tag{8}$$

We note that the path costs and travel disutilities are functions of the path-flow pattern f (a vector of $\{\ldots, f_p^{rs}, \ldots\}$). Therefore, the traffic equilibrium problem with variable demand is to find a path-flow pattern f^*, which induces a demand pattern $q^* = q(f^*)$ such that, for every OD pair (r,s) and each path $p \in P^{rs}$, the following conditions hold:

$$\eta_p^{rs}(f^*) - \pi^{rs}(q(f^*)) \begin{cases} = 0 & \text{if } f_p^{rs^*} > 0 \\ \geqslant 0 & \text{if } f_p^{rs^*} = 0 \end{cases}, \quad \forall r \in R,\ s \in S,\ p \in P^{rs}. \tag{9}$$

That is, when the travel cost on path p is larger than the travel disutility, the flow on that path is zero. When the travel cost on path p is equal to the travel disutility, its flow is greater than or equal to zero. These conditions are equivalent to Wardrop's Principle: all of the used paths have equal and minimum travel times; all of the unused paths have equal or higher travel times [21].

As observed by Aashtiani [22], the above equilibrium conditions are equivalent to the nonlinear complementarity problem (NCP):

$$x \geqslant 0, \quad F(x) \geqslant 0, \quad x^T F(x) = 0, \tag{10}$$

obtained by setting $x = f$ and letting $F(x) = \eta - \pi$, where η is a vector of $\{\ldots, \eta_p^{rs}, \ldots\}$ and π is a vector of $\{\ldots, \pi^{rs}, \ldots\}$. Aashtiani established that the above NCP is equivalent to the traffic equilibrium problem. The proof is for the travel demand function, but it is also valid for the travel disutility function adopted in this paper, which is the inverse of the travel demand function (see Chapter 4 in Nagurney [20] for details).

This NCP formulation offers the flexibility of relaxing the assumption of linear path costs, while including the linear path cost function as a special case – a feature used in most existing formulations. For purposes here, the principal benefit of this formulation is its ability to accommodate nonlinear path costs with multiple criteria.

3. Bicriteria shortest path problem and algorithm

A difficulty in solving the bicriteria shortest path problem (BCSPP) is that there may be no single optimal solution that satisfies both objectives simultaneously. If there were, the solution to the BCSPP would be straightforward because the best path will dominate all other paths in terms of both objectives. Furthermore, when the path cost function is nonlinear (or non-additive), conventional labeling-based algorithms may not be applicable because they may violate Bellman's Principle of Optimality. In other words, monotonicity may not hold and therefore no general efficient method exists to obtain the optimal path without first generating all non-dominated paths.

Here we describe an exact approach, based on Hansen's method, to generate the entire set of non-dominated paths. It extends the generic label-setting shortest path algorithm (such as Dijkstra algorithm) into a multiple-labeling scheme. For simplicity, we consider only two labels (travel time and toll), but the algorithm can be generalized to any number of labels. Consider that each link (i,j) has two-label vector $c_{ij} = (t_{ij}, \tau_{ij})^T$. Denote l_j^k as a two-label vector of the kth path from origin node r to node j; cc_j^{ki} a cost of ith attribute of the kth path from origin node r to node j; and R_j the set of indices i of the non-dominated temporary vector labels of node j. In addition, let T denote the set of nodes for which $R_j \neq \emptyset$ and V the set of indices for which $j \in T$ or R_j was $\neq\emptyset$. The algorithm is defined as follows:

STEP 1. Initialization

Set $l_r^1 = (0,0); l_j^1 = (\infty,\infty)$ for all nodes $j \neq O$; $T = \{r\}$, $R_r = \{1\}$, $V = \{1\}$, $R_j = \emptyset$ for all nodes $j \neq O$;

STEP 2. Selection of a node with the smallest vector labels

IF $T = \emptyset$ THEN

GOTO Step 4.

ELSE

Compute $c = \min\left\{cc_j^{i,1} | j \in T, i \in R_j\right\}$ and select node j such that $j = \max\left\{k \middle| cc_k^{ii,1} = c, k \in T, ii \in R_k\right\}$. Delete ii from R_j and j from T if $R_j = \emptyset$. If j is equal to the destination, STOP.

OTHERWISE

GOTO Step 3.

STEP 3. Computation of new labels

For each node k emanating from node j, calculate new vector labels of the ii-path (i.e., $l = l_j^{ii} + c_{jk}$) and do the followings:

IF $k \notin V$ THEN

Introduce a new vector labels for node k, set $R_k = \{1\}$, $T = T \cup \{k\}$, $V = V \cup \{k\}$. Return to Step 2.

ELSE IF $k \in V$ and $k \notin T$ THEN

Compare the new vector labels with the non-dominated vector labels of node k; if it is dominated, erase it; otherwise add it to the list; choose i the first value not yet used in R_k; set $R_k = \{i\}$, $T = T \cup \{k\}$. Return to Step 2.

ELSE IF $k \in V$ and $k \in T$ THEN

Compare the new vector labels with the non-dominated unselected vector labels of node k; if some of them are dominated by the new vector labels, erase them and delete their index from R_k. Then compare the new vector labels with all the non-dominated vector labels of node k and if it is dominated, erase it; otherwise, add it to the list; choose i the first value not yet used in R_k; set $R_k = R_k \cup \{i\}$. Return to Step 2.

STEP 4. Selection of optimal path

Since a path cost function is known, and all non-dominated paths sought for have been found from Step 2, it is trivial to select the optimal path based on the least cost value.

4. Projection and contraction method

Let Ω be a nonempty subset of R^n, and F be a monotone mapping to itself. The variational inequality problem, denoted as VI(Ω,F), is to find a vector $x^* \in \Omega$ such that

$$F(x^*)^T(x - x^*) \geqslant 0, \quad \forall x \in \Omega. \tag{11}$$

When $\Omega = \{x \in R^n | x \geqslant 0\} = R_+^n$, (11) can be reformulated as a nonlinear complementarity problem as follows:

$$x \geqslant 0, \quad F(x) \geqslant 0, \quad x^T F(x) = 0. \tag{12}$$

The nonlinear complementarity problem, denoted as NCP(R_+^n, F), is a special case of VI(Ω,F). Every solution of the NCP is also a solution for the VI (see [20]) for details). Thus, the projection and contraction methods developed for VI(Ω,F) are also applicable for NCP(R_+^n, F).

A basic property of projection mapping on a closed convex set (see [23] for details) is

$$(v - P_\Omega(v))^T (P_\Omega(v) - x) \geqslant 0, \quad \forall v \in R^n, \ \forall x \in \Omega, \tag{13}$$

which will be used later in the qualitative analysis of the PC methods. It is well known that NCP(R_+^n, F) is equivalent to the following *projection equation*:

$$x = P_\Omega[x - F(x)]. \tag{14}$$

Thus, solving the NCP(R_+^n, F) is equivalent to solving the non-smooth equation, i.e., finding a zero point of the residual of the projection equation

$$e(x) = x - P_\Omega[x - F(x)]. \tag{15}$$

In fact, $\|e(x)\|$ can be viewed as an error bound for NCP(R_+^n, F) that measures the deviation of x from Ω^*. Naturally, $\|e(x)\|_\infty$ can also be used as a stopping criterion to monitor the convergence.

Fundamental inequalities

Let $x^* \in \Omega^*$ be a solution to (14). For any $x \in R^n$,$P_\Omega[x - F(x)] \in \Omega$. It follows from (11) that

$$(\textbf{FI1}) \quad F(x^*)^T (P_\Omega[x - F(x)] - x^*) \geqslant 0, \quad \forall x \in R^n. \tag{16}$$

Setting $v = x - F(x)$ and $x = x^*$ in inequality (13) and using the notation $e(x)$, we obtain

$$(\mathbf{FI2}) \quad \{e(x) - F(x)\}^T (P_\Omega[x - F(x)] - x^*) \geqslant 0, \quad \forall x \in R^n. \tag{17}$$

Under the assumption that F is monotone, we have

$$(\mathbf{FI3}) \quad \{F(P_\Omega[x - F(x)]) - F(x^*)\}^T (P_\Omega[x - F(x)] - x^*) \geqslant 0, \quad \forall x \in R^n. \tag{18}$$

As the basis for the development of different PC algorithms, Inequalities (16)–(18) are called the three fundamental inequalities (FI), labeled here as FI1, FI2, and FI3, respectively, for easy reference. FI1 follows directly from the definition of variational inequalities; FI2 results from the basic property of projection mapping; and FI3 is based on the assumption of monotonicity of the mapping F.

As observed by He [16], the search directions of many projection and contraction methods are constructed based on the FI. If F is a monotone affine mapping, the search direction can be constructed based on either FI1 alone [24,25] or FI1 + FI2 together [26,27]. Both methods are simple minimization methods without line search and their implementations are simple. For a nonlinear monotone mapping F, the extra-gradient method by Korplelevich [28] and the extra-gradient method with Armijo's line search by Sun [29] use FI1 + FI3 to obtain the search direction. In this paper, we use all three fundamental inequalities to construct the search direction proposed by He [16] and independently discovered by Solodov and Tseng [30] and Sun [31]. By adding FI1 + FI2 + FI3, we obtain

$$\{e(x) - [F(x) - F(P_\Omega[x - F(x)])]\}^T \{(x - x^*) - e(x)\} \geqslant 0, \quad \forall x \in R^n. \tag{19}$$

Denote

$$d(x) = e(x) - \{F(x) - F(P_\Omega[x - F(x)])\}. \tag{20}$$

It follows from (19) that

$$(x - x^*)^T d(x) \geqslant e(x)^T d(x), \quad \forall x \in R^n. \tag{21}$$

For convenience, we first assume that mapping F is Lipschitz continuous with a constant$L \in [0,1)$, i.e.,

$$\|F(x) - F(P_\Omega[x - F(x)])\| \leqslant L\|e(x)\|, \quad \forall x \in R^n. \tag{22}$$

Under this assumption, we have

$$e(x)^T d(x) = \|e(x)\|^2 - e(x)^T \{F(x) - F(P_\Omega[x - F(x)])\} \geqslant \|e(x)\|^2 - \|e(x)\| \|F(x) - F(P_\Omega[x - F(x)])\|. \geqslant (1 - L)\|e(x)\|^2 \tag{23}$$

and via (21), it follows that

$$(x - x^*)^T d(x) \geqslant (1 - L)\|e(x)\|^2, \quad \forall x \in R^n. \tag{24}$$

This inequality is the foundation for constructing a contraction method. That is, because

$$\left(\nabla\left(\frac{1}{2}\|x - x^*\|^2\right)\right)^T d(x) \leqslant -(1 - L)\|e(x)\|^2, \quad \forall x \in R^n. \tag{25}$$

In other words, $-d(x)$ is a descent direction that minimizes the error of $\|x - x^*\|$. Although the solution point x^* is unknown, we can find a new iterate, which reconstruct along the descent direction $-d(x)$ to yield a new point that reduces the value of the distance function $\frac{1}{2}\|x - x^*\|^2$. This new iterate is a better approximate than the current point x. Thus, the sequence $\{x^n\}$ generated by the projection and contraction method using FI1 + FI2 + FI3 as the search direction is convergent. Because a projection is made in every iteration and the Euclidean distance of the iterates to the solution set monotonically contracts to zero, the method is called projection and contraction. See He [26, Theorem 3], for a detailed proof of convergence of the PC method. Here we only sketch out the main ideas underlying the PC method.

For a general continuous monotone mapping F, the Lipschitz assumption may not be satisfied. Note that the NCP is invariant under the multiplication of F by some positive scalar β. We denote

$$e(x, \beta) = x - P_\Omega[x - \beta F(x)] \tag{26}$$

and

$$d(x, \beta) = e(x, \beta) - \beta\{F(x) - F(P_\Omega[x - \beta F(x)])\}. \tag{27}$$

It follows that

$$(x - x^*)^T d(x, \beta) \geqslant e(x, \beta)^T d(x, \beta), \quad \forall x \in R^n. \tag{28}$$

Because the mapping F is continuous, we can find a small enough $\beta > 0$, such that

$$\|\beta F(x) - \beta F(P_\Omega[x - \beta F(x)])\| \leqslant L\|e(x, \beta)\|. \tag{29}$$

Similar to (23) and (24), we have

$$(x-x^*)^T d(x,\beta) \geqslant e(x,\beta)^T d(x,\beta) \geqslant (1-L)\|e(x,\beta)\|^2. \tag{30}$$

In practice, we use an additional procedure to get a suitably scaled F satisfying (22) before constructing the search direction. This is accomplished by using an adaptive scaling procedure to find a suitable β. The self-adaptive projection and contraction algorithm is given as follows:

Self-adaptive projection and contraction (SA-PC) algorithm

Step 0. Let $\varepsilon > 0$, $\gamma \in (0,2)$, and $\beta_0 = 1$. Given $x^0 \in \Omega$ and set $n := 0$.
Step 1. WHILE $\|\beta_n F(x^n) - \beta_n F(P_\Omega[x^n - \beta_n F(x^n)])\|^2 > 0.95\|e(x^n,\beta_n)\|^2$
DO $\beta_n := \beta_n * \sqrt{2}$ ENDIF.

$$x^{n+1} := x^n - \gamma d(x^n, \beta_n).$$

Step 2. IF $\|\beta_n F(x^n) - \beta_n F(P_\Omega[x^n - \beta_n F(x^n)])\|^2 < 0.20\|e(x^n,\beta_n)\|^2$
THEN $\beta_{n+1} := \beta_n/\sqrt{2}$, ELSE $\beta_{n+1} := \beta_n$ ENDIF.
Step 3. IF $\|e(x^n,\beta_n)\|_\infty \leqslant \varepsilon$, THEN terminate.
OTHERWISE, set $n := n + 1$ and GOTO Step 1.

From the above solution procedure, it should be noted that only two function evaluations and a simple projection on the non-negative orthant are required in each iteration. The self-adaptive stepsize rule allows the sequence β_n to be non-monotone (i.e., β_n can decrease as well as increase to fulfill the Lipschitz condition). Furthermore, the global convergence can be shown under the monotone assumption on the underlying mapping F without the need to know the Lipschitz constant L in advance. For more details about the self-adaptive stepsize updating scheme and the convergence properties of the *SA-PC* algorithm, we refer to He [16], Solodov and Tseng [30], and Sun [31].

5. Implementation of bicriteria traffic equilibrium problem with variable demand and nonlinear path costs

Based on the BCSSP and PC method discussed above, we now can assemble the two components together to solve the bicriteria traffic equilibrium problem with variable demand and nonlinear path costs. At each iteration, Hansen's method is used to generate optimal paths using a nonlinear path cost function that combines time and toll. This step acts as a column generation procedure to automatically generate paths in each iteration as needed. Then, the self-adaptive PC method is used to solve the equilibration problem by distributing flows to paths such that the Wardrop's principle is satisfied. We also mention that the computational effort required for the PC method is very modest. It consists of a trivial projection onto the non-negative orthant and two evaluations of the mapping F. Recall that our mapping F for the bicriteria traffic equilibrium problem with variable demand and nonlinear path costs is

$$F(x) = \eta(f) - \pi(q), \tag{31}$$

where $\eta(f)$ and $\pi(q)$ are the path cost functions and travel disutility functions, respectively. For a given path $p \in P^{rs}$ between OD pair (r,s), the corresponding component of $F(x)$ to f_p^{rs} is given as

$$F_p^{rs}(x) = \eta_p^{rs}(f) - \pi^{rs}(q). \tag{32}$$

Similarly, the component of $e(x)$ is

$$e_p^{rs}(x) = f_p^{rs} - P_\Omega[f_p^{rs} - F_p^{rs}(x)]. \tag{33}$$

The detailed algorithmic steps are described as follows:

Step 0. *Initialization*: Start with free-flow travel times
0.1 Set $\kappa > 0$, $\varepsilon > 0$, $v \in (0,0.5)$, and $m = 0$.
0.2 Perform incremental assignment to generate an initial set of paths: $P^{rs}(m)$, $\forall r \in R$, $s \in S$.
Step 1. Column generation
1.1 Update link travel times and path costs: η_p^{rs}, $\forall r \in R, s \in S, p \in P^{rs}(m)$.
1.2 Solve the *bicriteria* shortest path problem: μ^{rs} and $\bar{p}^{rs}$, $\forall r \in R, s \in S$, where μ^{rs} is the optimal path cost resulting from solving the BCSSP and $\bar{p}^{rs}$ is the arc sequence denoting the optimal path.
Step 2. *Convergence*
2.1 IF $\max_{rs}\sum_p \frac{f_p^{rs}}{q^{rs}}\left(\frac{\eta_p^{rs}-\mu^{rs}}{\eta_p^{rs}}\right) \leqslant \kappa$, THEN terminate.
2.2 OTHERWISE, set $m = m + 1$, update path set: $P^{rs}(m) = \bar{p}^{rs} \cup P^{rs}(m-1)$ IF $\bar{p}^{rs} \notin P^{rs}(m-1)$, $\forall r \in R, s \in S$, reduce $\varepsilon = \varepsilon \cdot v$, and GOTO the equilibration procedure.
Step 3. *Equilibration*
3.1 Use SA-PC algorithm to solve the NCP using ε as the terminating threshold and path set $P^{rs}(m)$, $\forall r \in R$, $s \in S$.

3.2 Drop unused paths: if $f_p^{rs} = 0$, then $P^{rs}(m) = P^{rs}(m) - p, \quad \forall r \in R, s \in S, p \in P^{rs}(m)$.
3.3 Return to Step 1.

Remark 1. In the initialization procedure, we adopted an incremental assignment procedure (see [18] for details) to incrementally generate an initial set of paths. This technique is typically used in traffic assignment to create a good initial path set.

Remark 2. In the column generation procedure, we used the Hansen's method as an exact method to generate the entire set of non-dominated paths. It may not be an efficient method since the number of non-dominated paths grows exponentially with the number of criteria and/or network size. Work is currently in progress to develop an approximation method using piecewise linearization and branch and bound techniques to avoid generating the entire set of non-dominated paths.

Remark 3. In the equilibration procedure, the convergence error ε is progressively reduced to provide further efficiency. This idea is also implemented in the disaggregate simplicial decomposition algorithm for solving the traffic assignment problem [32]. In addition, unused paths (i.e., $f_p^{rs} = 0$) are dropped to keep the path set compact.

6. Numerical tests

We test the proposed algorithm on two test networks. The basic data for these two networks are summarized in Table 1. Network characteristics for network 1 are provided in Fig. 1. A one-unit toll was imposed on links 2 → 3 and a two-unit toll was added on link 3 → 6. The links with toll are highlighted in Fig. 1. Network 2 is the classical Sioux Falls network provided in Fig. 2. Link characteristics can be found in LeBlanc et al. [33].

For both networks, we adopt the standard Bureau of Public Road (BRP) as the link cost function:

$$t_a = \alpha_a\left(1 + 0.15\left(\frac{v_a}{c_a}\right)^4\right), \tag{34}$$

where t_a, α_a, v_a, and c_a are the travel time, free-flow travel time, flow, and capacity on link a, respectively. The demands are elastic with known OD travel disutility functions of the following functional form:

$$\pi_{rs}(q) = -m_{rs}q_{rs} + h_{rs}. \tag{35}$$

The numerical tests are not only aim at demonstrating the computational efficiency, but also at verifying the validity of the algorithm and examining the differences between using travel time as the sole criterion in route selection and incorporating a second criterion (e.g., toll) to assess the tradeoff in both linear and nonlinear path costs.

For network 1, we use the following three path costs for the comparison:

(i) Single-criterion linear (SCL) path cost function

$$\eta_p^{rs} = \sum_{a\in A} \delta_{pa}^{rs} t_a. \tag{36}$$

(ii) Bicriteria linear (BCL) path cost function

$$\eta_p^{rs} = 1.2\left(\sum_{a\in A} \delta_{pa}^{rs} t_a\right) + \sum_{a\in A} \delta_{pa}^{rs} \tau_a. \tag{37}$$

(iii) Bicriteria nonlinear (BCN) path cost function

$$\eta_p^{rs} = 2.0\left(\frac{\sum_{a\in A}\delta_{pa}^{rs} t_a}{10}\right)^2 + \left(\frac{\sum_{a\in A}\delta_{pa}^{rs} t_a}{10}\right) + 3.0\left(\sum_{a\in A} \delta_{pa}^{rs} \tau_a\right). \tag{38}$$

In order to have a meaningful comparison among the three path cost models, parameter m_{rs} is fixed at the same value, while parameter h_{rs} is individually adjusted so that all three models produce approximately the same level of OD travel demand. The parameters m_{rs} and h_{rs} used in the travel disutility function for network 1 are provided in Table 2.

Table 1
Data for test networks.

	Network 1	Network 2
Nodes	9	24
Links	12	76
OD pairs	1	528

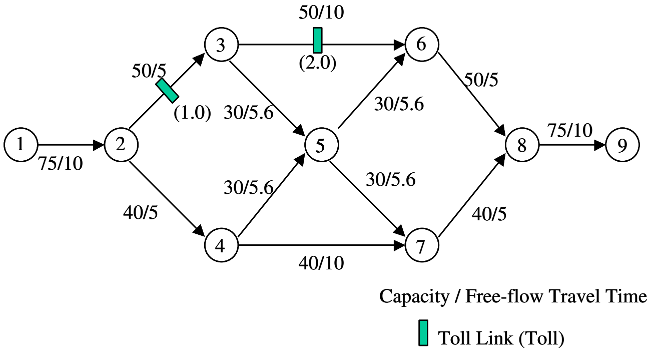

Fig. 1. Test network 1.

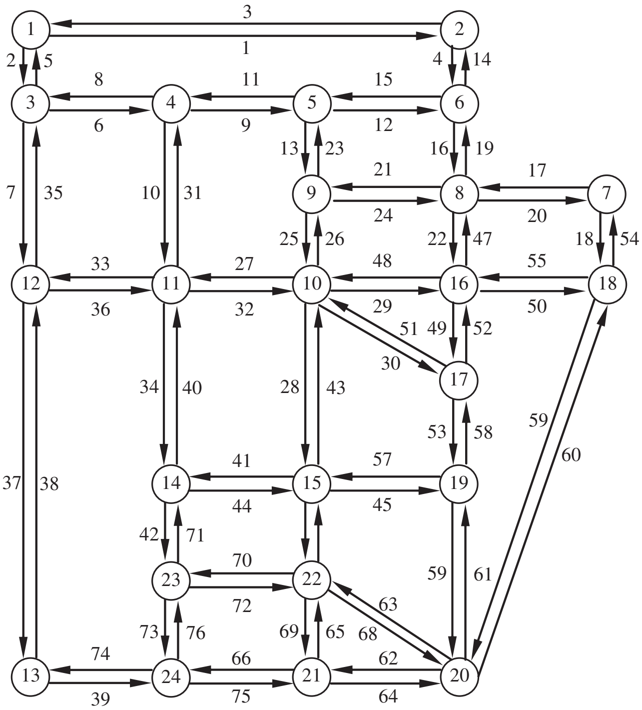

Fig. 2. Test network 2.

The complete link-flow patterns of the three path cost functions are shown in Fig. 3. The numbers in the figure represent the three link-flow patterns resulting from the SCL, BCL, BCN path cost models, respectively. As can be seen from Table 3, although the total demands remain relatively constant for all three path cost models, the link-flow patterns are different. Both BCL and BCN assign less traffic on the two toll links, particularly on link 3 → 6 that has a higher toll, and between these two models BCN assigns even less traffic compared to BCL. This reflects the tradeoff between single-criterion and bicriteria route choices as well as the differences between BCL and BCN path costs.

For completeness, we also provide the path-flow patterns in Table 3. One should be careful when comparing path-flow solutions since they are generally not unique. A different set of equilibrium path flows could have been generated if a different initial solution were used. However, Table 3 provides useful information that can be used to check the correctness of

Table 2
Travel disutility parameters.

	SCL	BCL	BCN
m_{rs}	0.25	0.25	0.25
h_{rs}	100.00	89.35	91.14

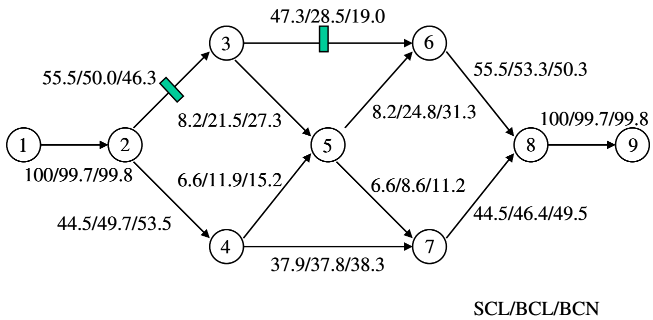

Fig. 3. Comparison of link flows. (The three numbers denote SCL, BCL, and BCN link flows.)

the assignment results. First, the path flows sum to up to the OD's travel demand. Second, the costs on all used paths between the OD pair are equal to its travel disutility for all three path cost models. These two factors demonstrate that the solution is valid.

At termination, the algorithm found 4, 5, and 5 used paths for SCL, BCL, and BCN, respectively. Three of the used paths are common to all three path cost models, but the flow allocations to these paths are different. The total flow allocations to the common paths are 93.41, 79.25, and 73.41 for SCL, BCL, and BCN, respectively. When tolls are considered in the route selection, both BCL and BCN found two other used paths as shown in Table 3. Because the path cost functions are different, the flow allocations are also different. These results basically show that adopting a bicriteria route choice with different functional path costs leads to different results that may be useful for describing drivers' route choice behaviors.

Table 3
Comparison of used path flows.

Model	Demand	Travel disutility	Path (node sequence)	Path flow	Path cost
SCL	100	52.97	1–2–3–6–8–9	47.37	52.98
			1–2–4–7–8–9	37.88	52.97
			1–2–3–5–6–8–9	8.16	52.98
			1–2–4–5–7–8–9	6.59	52.97
BCL	99.73	64.51	1–2–3–6–8–9	28.45	64.50
			1–2–4–7–8–9	37.87	64.50
			1–2–3–5–6–8–9	12.93	64.49
			1–2–4–5–6–8–9	11.89	64.51
			1–2–3–5–7–8–9	8.60	64.51
BCN	99.75	65.56	1–2–3–6–8–9	18.96	65.56
			1–2–4–7–8–9	38.31	65.55
			1–2–3–5–6–8–9	16.14	65.55
			1–2–4–5–6–8–9	15.19	65.56
			1–2–3–5–7–8–9	11.16	65.56

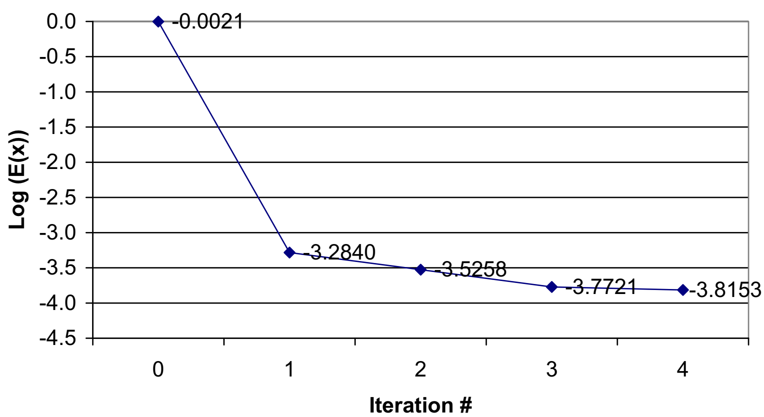

Fig. 4. The logarithm of residual error of BCN path cost model for network 1.

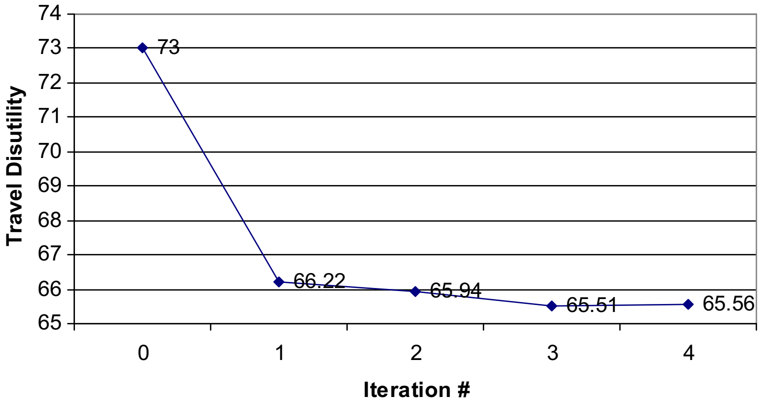

Fig. 5. OD travel disutility of BCN path cost model for network 1.

The convergence behavior of the algorithm, which is given in terms of the logarithm of $e(x)$, is provided in Fig. 4. The residual error reported here is for the BCN path cost model at each iteration before the equilibration procedure begins. As can be seen, the algorithm quickly finds the zero point of the error bound within four iterations. For each iteration, we use the PC method to repeatedly solve the equilibration procedure until it satisfies its terminating criterion (see Step 3.1). Similarly, the trajectory of the OD travel disutility is depicted in Fig. 5.

A measure of the computational performance of the algorithm is provided in Tables 4 and 5, as represented by the cumulative number of inner iterations and cumulative number of evaluations of mapping F given in (32), for different stopping accuracies as well as different starting scaling factor, respectively. The results show that the algorithm is quite robust in achieving very accurate solution and is insensitive to the initial scaling factor. Using the BCN path cost model, Table 6 further shows the variable demand as a function of travel disutility for network 1. As the tolls on link $2 \rightarrow 3$ and link $3 \rightarrow 6$ increase, travel disutility increases which in return lowers the travel demand.

For network 2, we present only the convergence results to demonstrate that the proposed algorithm is also applicable to medium-sized networks. Fig. 6 shows the logarithm of $e(x)$ at each iteration. It starts with a huge error (i.e., $\log(5333) = 3.727$) at iteration 0, but quickly reduces in the next iteration and finds the zero point within five iterations.

Table 4
Computational performance with different stopping accuracies.

Error	0.1	0.01	0.001	0.0001	0.00001	0.000001
# of inner iterations	362	864	865	1065	1351	1538
# of function evaluations	385	958	959	1186	1522	1740

Table 5
Computational performance with different starting scaling factors.

Initial β	0.01	0.1	1.0	2.0	5.0	10.0	20.0
# of inner iterations	881	888	865	859	889	880	877
# of function evaluations	955	925	959	958	956	975	983

Table 6
Variable demand of BCN path cost model for network 1.

Tolls		Travel disutility	Travel demand
Link 2–3	Link 3–6		
0.5	1.0	65.01	101.94
1.0	2.0	65.56	99.75
1.5	3.0	66.35	96.59
2.0	4.0	67.00	94.01
3.0	6.0	68.89	93.89

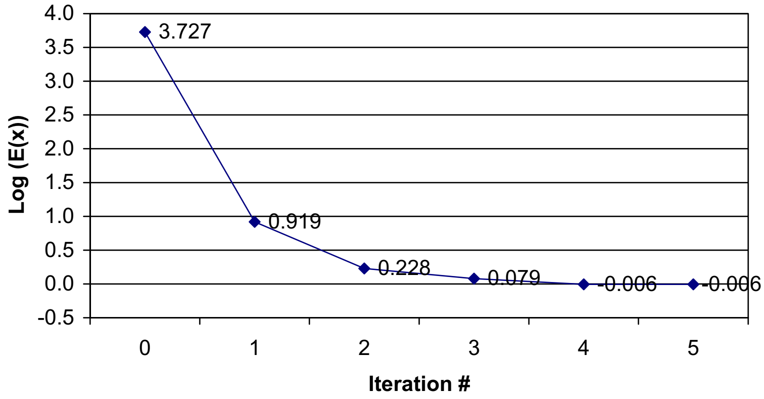

Fig. 6. The logarithm of residual error of BCN path cost model for network 2.

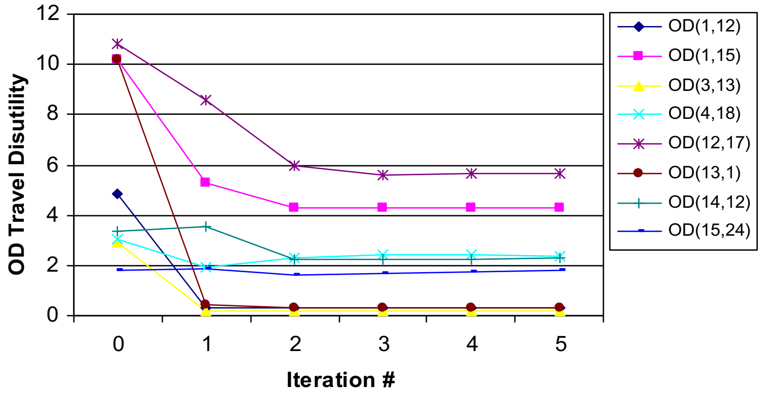

Fig. 7. OD travel disutility of BCN path cost model for network 2.

We also show the trajectories of the travel disutility for a few randomly selected OD pairs in Fig. 7. Similar to the residual error, the OD travel disutilities converge in five iterations.

7. Conclusions

In this paper, we have presented an algorithm for solving the bicriteria traffic equilibrium problem with variable demand and nonlinear path costs. The algorithm generates nonlinear cost paths, as needed, using a bicriteria shortest path algorithm, and equilibrates path flows via a projection and contract (PC) method. The main advantages of the PC method are simplicity and ability to handle a general monotone mapping F. Differentiability of mapping F is not required. The computational effort required per iteration is very modest. It consists of two function evaluations to construct the search direction and a simple projection on the non-negative orthant. The scaling factor β is self-adaptive in the sense that it automatically adjusts to satisfy the Lipschitz condition. Initial results indicate that the algorithm is capable of solving a class of traffic equilibrium problem with multi-dimensional nonlinear path costs.

Although the algorithm is capable of being extended to any number of criteria, more work needs to be performed on larger networks to demonstrate the efficiency of the algorithm, particularly relative to the shortest path routine with multiple criteria. Since the number of non-dominated paths may grow exponentially with the number of criteria and/or network size, an efficient approximation method should be developed to avoid generating the entire set of non-dominated paths. Work is currently in progress to develop an approximation method using piecewise linearization and branch and bound techniques.

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